EXAMPLES OF 2-UNRECTIFIABLE NORMAL CURRENTS

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ABSTRACT. We construct new examples of normal (metric) currents using inverse systems of cube complexes. For any $N \geq 2$ we provide examples of N-dimensional normal currents whose associated vector fields are simple, and whose supports are purely 2-unrectifiable and have Nagata dimension N. We show that in l^∞ normal currents can be realized as limits in the flat distance of currents associated to cube complexes.

CONTENTS

1. INTRODUCTION

Background. Metric currents were introduced by Ambrosio-Kirchheim in [AK00a] to generalize the notion of Federer-Fleming [FF60] currents to the metric setting. One motivation to introduce these objects is formulating / understanding Plateau's problem in metric spaces; there are also other geometric applications, see for instance [Wen06, Wen11a, Wen11b]. Lang [Lan11] has also formulated a more general version of the theory in [AK00a] which is more suitable for some geometric applications as it does not require the mass of currents to be finite. In the following we often use the word current to refer to metric currents, when we refer to Federer-Fleming currents we always use the term classical current.

Unfortunately, as of today there are not many examples of metric currents. Most of the theory / applications has been developed looking either at *rectifiable* or *integral currents*, as these objects admit an alternative (and more concrete) description in the framework of *rectifiable sets* [AK00b] in metric spaces. Thus, this work was in part motivated by the wish to provide new examples of metric currents.

Moreover, it is not even clear what metric currents in Euclidean spaces are; while the Ambrosio-Kirchheim normal currents coincide in $\mathbb{R}^n$ with the classical

normal currents (of finite mass), for general currents it is only known that the 1-dimensional ones are flat [Sch14a]. While [Sch14a] gives a geometric description of 1-currents in metric spaces, when the first version of this preprint appeared even the specific question of whether 2-dimensional normal currents in $\mathbb{R}^4$ have some special geometric structure was open.

In the setting of Carnot groups Williams [Wil12] has obtained a complete classification of normal currents, and also a partial classification of general metric currents. The study of Carnot groups actually shows a drawback in the existing definition of metric currents. In fact, in a non-abelian Carnot group $\mathbb{G}$ there are many objects which satisfy all the axioms in [AK00a, Lan11] except for the *joint continuity* (Axiom (ii) in Def. 3.1 in [AK00a]; Axiom (2) in Def. 2.1 in [Lan11]). Concretely, a k-dimensional "current" T might be obtained using an integral representation (like for a classical current):

$$(1.1) \qquad\qquad T = \vec{T}\,\mu_{\mathbb{G}},$$

$\vec{T}$ being a smooth k-vector field (in the k-th exterior power of the horizontal distribution), and $\mu_{\mathbb{G}}$ denoting the Haar measure (here we use Lang's definition, otherwise just restrict $\mu_{\mathbb{G}}$ to a set of finite measure). In general, T is not going to satisfy the joint continuity axiom: for example, in the first Heisenberg group $X \wedge Y\,\mathcal{L}^3$ would not define a 2-current in the sense of [AK00a, Lan11].

Williams [Wil12] has also provided examples of 2-dimensional normal currents in purely 2-unrectifiable Carnot groups. These examples are of the form $(\vec{T}_1 - \vec{T}_2)\,\mu_{\mathbb{G}}$ where $\vec{T}_1$ and $\vec{T}_2$ are constant simple 2-vectors chosen to cancel their "boundaries". Note again that the "currents" $\vec{T}_1\,\mu_{\mathbb{G}}$, $\vec{T}_2\,\mu_{\mathbb{G}}$ are not actually currents in the sense of [AK00a, Lan11]: this is unavoidable as [Wil12] shows that a k-unrectifiable Carnot group cannot admit a k-normal current where the vector field is simple.

The arguments in [Wil12] also show that it is convenient to work with an integral representation of metric currents. In [Sch14a] we have showed that any k-current T, in the sense of [AK00a, Lan11], and whose support is a countable union of doubling metric spaces, admits an integral representation:

$$(1.2) \qquad\qquad T = \vec{T}\,\|T\|$$

where $\|T\|$ is the mass measure, and $\vec{T}$ is a k-vector field in the sense of Defn. 2.23. While the representation (1.2) is analogous to the classical setting, a T satisfying (1.2) is not necessarily going to be a current in the sense of [AK00a, Lan11], as it might fail the joint continuity axiom. Having in mind the case of Carnot groups, in this work we will *drop the joint continuity axiom* from the definition of metric current, i.e. we will define a k-current T to be an object admitting a representation like (1.2): these objects were called *precurrents* in [Sch14a] following a terminology introduced by [Wil12]. Note that by Theorem 5.35 in [Sch14a] the normal currents in the sense of [AK00a, Lan11] and in our extended sense *coincide*; as in this work we are essentially concerned with normal currents, dropping the joint continuity axiom does not cause any inconsistency. However, our more general notion of current might be of independent interest, e.g. in the setting of Carnot groups.

1.1. **Results.** The Question which motivated this paper is:

(**Q1**): Are there examples of nontrivial normal N-dimensional currents ($N \geq 2$) whose N-vector field is simple and whose support is purely N-unrectifiable?

In Section 4 we

(Ex): Exhibit for each $N \geq 2$ an example N_∞ of a nontrivial normal N-dimensional current whose support X_∞ is purely 2-unrectifiable and has Nagata and topological dimension N. Moreover, the metric measure space $(X_\infty, \|N_\infty\|)$ may be taken to admit a $(1,1)$-Poincaré inequality.

These examples are optimal from three perspectives:

(Prs1): One cannot have a 1-unrectifiable support because of [PS12, PS13];

(Prs2): By a result of Züst [Züs11] if a space X supports a nontrivial normal k-current it must have Nagata dimension at least k (even though the topological dimension might be 1, but not 0 by [PS12, PS13]);

(Prs3): The vector field is simple in contrast to the case of Carnot groups [Wil12].

Essentially, the Nagata dimension [LS05] is a version of the topological dimension in the Lipschitz category, and thus it is better suited to handle some questions arising in analysis on metric spaces, e.g. questions regarding the extendability of Lipschitz maps. Note that the topological dimension always bounds from below the Nagata dimension. By a beautiful Theorem of Buyalo-Lebedeva [BL07] the Nagata dimension and the topological dimension of a *self-similar* metric space coincide. As an application one concludes that the Nagata dimension of a Carnot group coincides with the topological dimension (Carnot groups are self-similar because they have a familiy of dilations and translations; for another argument see [LDR15]). Note also that by [Wil12] an n-dimensional non-abelian Carnot group cannot admit a nontrivial n-normal current.

The examples N_∞, X_∞ are obtained by relaxing the requirements of inverse systems $\{(X_i, \mu_i)\}_i$ in [CK15] (see Defn. 3.14). In Theorem 3.20 we show how to associate a limit current to the inverse limit (X_∞, μ_∞) of such a system and also show that (X_∞, μ_∞) admits a kind of "calculus" similar to the one in $\mathbb{R}^N$ (despite being, in general, purely 2-unrectifiable).

The spaces $\{X_i\}$ are N-dimensional cube complexes and the measures $\{\mu_i\}$ restrict to a constant multiple of Lebesgue measure on each cell. Moreover, to each X_i one can naturally associate a normal "cubical" current N_i, and N_∞ is the weak limit of the N_i. Therefore, a natural question is how general is the idea of constructing a normal N-current as a limit of cubical currents. In Section 5 we show that in l^∞ one can approximate normal currents by cubical ones in the flat distance (and hence in the weak topology) while keeping good bounds on the masses.

Finally, in analyzing the module of Weaver derivations for these examples we have found useful some results relating the Nagata dimension and approximations of X by polyhedra, see Subsection 2.3.

1.2. **Further directions.** In the first version of this work we asked the following natural question:

(Q2): Is it possible to construct in $\mathbb{R}^{n \geq 4}$ a 2-dimensional nontrivial normal current N whose support is purely 2-unrectifiable?

G. Alberti had told us that such examples do not exist in $\mathbb{R}^3$. More generally a result of G. Alberti and A. Masacesi shows that in $\mathbb{R}^k$ a *codimension one* normal current can be represented as an integral of $(k-1)$-rectifiable currents. In a forthcoming paper, in joint work with U. Lang, we settle **(Q2)** in the negative, the crucial point being that **(Q2)** is about a *codimension two* current. Another natural question is:

(Q3): If (X, μ) is a metric measure space where X has Nagata dimension N, is the analytic dimension (Defn. 2.11) of (X, μ) at most N?

We provide a positive answer to **(Q3)**, Theorem 2.79, under an additional assumption. Note that by Theorem 2.55 a counterexample to **(Q3)** would provide for some $m \geq 1$ a Lipschitz map:

$$(1.3) \qquad F : X \to \mathbb{R}^{N+m}$$

with dF having rank $N + m$ on a set of positive measure, and such that F can be approximated in the weak* topology (i.e. pointwise with uniform bound on the Lipschitz constants) by maps:

$$(1.4) \qquad \tilde{F} : X \to \mathbb{R}^{N+m}$$

which factor through N-dimensional polyhedra. In particular, $d\tilde{F}$ would have rank at most N μ-a.e., while being close to dF in the weak* topology.

Acknowledgements. The examples discussed here arose in conversations between B. Kleiner and me, where we were looking at the question of producing purely 2-unrectifiable higher dimensional versions of the so-called Laakso spaces. I gratefully thank B. Kleiner for allowing me to include them here, and for motivating me to prove Theorem 5.3.

2. Background

In this Section we recall material on Weaver derivations, Metric currents, and the Nagata dimension. For Weaver derivations and metric currents we recall many concepts in a dry and formulaic style, and refer the interested reader to [Sch16, Sch14a, Sch14b] for more details. For the Nagata dimension we focus on new results on approximations by polyhedra (Theorem 2.55), and finite dimensionality (Theorem 2.79).

2.1. **Weaver derivations.** For more information we refer the reader to [Wea00, Sch16]. An $L^\infty(\mu)$**-module** M is a Banach space M which is also an $L^\infty(\mu)$-module and such that for all $(m, \lambda) \in M \times L^\infty(\mu)$ one has:

$$(2.1) \qquad \|\lambda m\|_M \leq \|\lambda\|_{L^\infty(\mu)} \|m\|_M.$$

Among $L^\infty(\mu)$-modules a special rôle is played by $L^\infty(\mu)$**-normed modules**:

Definition 2.2 (Normed modules)**.** An $L^\infty(\mu)$-module M is said to be an $L^\infty(\mu)$**-normed module** if there is a map

$$(2.3) \qquad |\cdot|_{M,\mathrm{loc}} : M \to L^\infty(\mu)$$

such that:

(1) For each $m \in M$ one has $|m|_{M,\mathrm{loc}} \geq 0$;
(2) For all $c_1, c_2 \in \mathbb{R}$ and $m_1, m_2 \in M$ one has:

$$(2.4) \qquad |c_1 m_1 + c_2 m_2|_{M,\mathrm{loc}} \leq |c_1||m_1|_{M,\mathrm{loc}} + |c_2||m_2|_{M,\mathrm{loc}};$$

(3) For each $\lambda \in L^\infty(\mu)$ and each $m \in M$, one has:

$$(2.5) \qquad |\lambda m|_{M,\mathrm{loc}} = |\lambda|\,|m|_{M,\mathrm{loc}};$$

(4) The local seminorm $|\cdot|_{M,\mathrm{loc}}$ can be used to reconstruct the norm of any $m \in M$:

$$\|m\|_M = \| \, |m|_{M,\mathrm{loc}} \, \|_{L^\infty(\mu)}. \tag{2.6}$$

Let $\mathrm{Lip_b}(X)$ denote the algebra of bounded real-valued Lipschitz functions defined on X. This is a Banach algebra with the norm the max of the sup norm $\|f\|_\infty$ and the Lipschitz constant $\mathbf{L}(f)$ (see [Wea99, Sec. 1.6]).

Definition 2.7 (Weaver derivation)**.** A **derivation** $D : \mathrm{Lip_b}(X) \to L^\infty(\mu)$ is a weak* continuous, bounded linear map satisfying the product rule:

$$D(fg) = f Dg + g Df. \tag{2.8}$$

A sequence $f_n \to f$ in weak* topology on $\mathrm{Lip_b}(X)$ if there is a uniform bound on the global Lipschitz constants $\mathbf{L}(f_n)$ of the $\{f_n\}$ and if $f_n \to f$ pointwise.

The collection of all derivations $\mathfrak{X}(\mu)$ is an $L^\infty(\mu)$-normed module [Wea00] and the corresponding local norm will be denoted by $|\cdot|_{\mathfrak{X}(\mu),\mathrm{loc}}$. Note also that $\mathfrak{X}(\mu)$ depends only on the measure class of μ.

Example 2.9. Consider $(\mathbb{R}^n, \mathcal{L}^n)$ and let ∂_α be the partial derivative in the α-direction. By Rademacher's Theorem ∂_α defines a bounded linear map $\partial_\alpha : \mathrm{Lip_b}(\mathbb{R}^n) \to L^\infty(\mathcal{L}^n)$. The weak* continuity can be reduced to the 1-dimensional case (and hence to integration by parts) using Fubini's Theorem.

Definition 2.10 (Submodules and locality)**.** Consider a Borel set $U \subset X$ and a derivation $D \in \mathfrak{X}(\mu \llcorner U)$. The derivation D can be also regarded as an element of $\mathfrak{X}(\mu)$ by extending Df to be 0 on $X \setminus U$. In particular, the module $\mathfrak{X}(\mu \llcorner U)$ can be naturally identified with the submodule $\chi_U \mathfrak{X}(\mu)$ of $\mathfrak{X}(\mu)$.

By Lemma 27 in [Wea00] derivations are **local** in the following sense: if U is μ-measurable and if $f, g \in \mathrm{Lip_b}(X)$ agree on U, then for each $D \in \mathfrak{X}(\mu)$, $\chi_U Df = \chi_U Dg$. Note that locality allows to extend the action of derivations on Lipschitz functions f, so that Df is well-defined.

Definition 2.11 (Analytic dimension)**.** We define the **analytic dimension** of the metric measure space (X, μ) to be the **index** of $\mathfrak{X}(\mu)$:

$$\text{index of } \mathfrak{X}(\mu) = \sup\{n \in \mathbb{N} : \exists U \text{ Borel: } \mathfrak{X}(\mu \llcorner U) \text{ contains } n\text{-independent} \tag{2.12}$$
$$\text{elements (over } L^\infty(\mu \llcorner U)) \text{ }\}.$$

If either $\mathrm{spt}\,\mu$ or X are doubling (see [Sch16]) then $\mathfrak{X}(\mu)$ has finite index; moreover, if $\mathfrak{X}(\mu)$ has finite index, it can be decomposed into a direct sum of free submodules (over smaller rings), see [Sch14b].

Example 2.13 (Index in $\mathbb{R}^n$). Let μ be a Radon measure on $\mathbb{R}^n$; by the Stone-Weierstrass Theorem for Lipschitz algebras [Wea99, Thm. 4.1.8] the polynomials in $\{x_1, \cdots, x_n\}$ (after truncating the polynomials to have bounded absolute value, e.g. postcomposing with $\max(\cdot, K)$, $\min(\cdot, K)$) are weak*-dense in $\mathrm{Lip_b}(\mathbb{R}^n)$ and so the index of $\mathfrak{X}(\mu)$ is at most n.

A standard way to produce derivations is to use Alberti representations. We deal here with a more restrictive situation, for the general case see [Sch16].

Definition 2.14 (Alberti representations)**.** Let $\mathrm{Curves}(X)$ denote the space of Lipschitz curves in X topologized with the Fell topology [Kec95, (12.7)] on their

graphs. Let μ be a Radon measure on X. An **Alberti representation** of μ is a pair $\mathcal{A} = [Q, w]$ where Q is a Radon measure on $\mathrm{Curves}(X)$ and w a Borel function $w : X \to [0, \infty)$ such that:

$$(2.15) \qquad \mu = \int_{\mathrm{Curves}(X)} w \cdot \mathcal{H}^1_\gamma \, dQ(\gamma),$$

where $\mathcal{H}^1_\gamma$ is the length measure on γ, and the integral is interpreted in the weak* sense. We say that $\mathcal{A}$ is C**-Lipschitz** (resp. $[C, D]$**-biLipschitz**) if Q is concentrated on the set of C-Lipschitz (resp. $[C, D]$-biLipschitz) curves.

Let $\mathcal{A} = [Q, w]$ be a C-Lipschitz Alberti representation of a measure $\nu \ll \mu$; then the formula:

$$(2.16) \quad \int_X g D_\mathcal{A} f \, d\nu = \int_{\mathrm{Curves}(X)} dQ(\gamma) \int wg \cdot \partial_\gamma f \, d\mathcal{H}^1_\gamma$$

$$(\forall (g, f) \in C_c(X) \times \mathrm{Lip}_\mathrm{b}(X)),$$

where $\partial_\gamma f$ denotes the derivative of f along γ, defines a derivation $D_\mathcal{A} \in \mathcal{X}(\nu) \subset \mathcal{X}(\mu)$ with $\|D_\mathcal{A}\|_{\mathcal{X}(\mu)} \leq C$.

Definition 2.17 (Weaver differentials)**.** The **module of 1-forms** $\mathcal{E}(\mu)$ is the dual module of $\mathcal{X}(\mu)$, i.e. it consists of the bounded module homomorphisms $\mathcal{X}(\mu) \to L^\infty(\mu)$. The module $\mathcal{E}(\mu)$ is an $L^\infty(\mu)$-normed module and the local norm will be denoted by $|\cdot|_{\mathcal{E}(\mu),\mathrm{loc}}$.

To each $f \in \mathrm{Lip}_\mathrm{b}(X)$ one can associate the 1-form, its **differential** $df \in \mathcal{E}(\mu)$, by letting:

$$(2.18) \qquad \langle df, D \rangle = Df \quad (\forall D \in \mathcal{X}(\mu));$$

the map $d : \mathrm{Lip}_\mathrm{b}(X) \to \mathcal{E}(\mu)$ is a weak* continuous 1-Lipschitz linear map satisfying the product rule $d(fg) = g df + f dg$. Note that by locality (Defn. 2.10) one can extend the domain of d to the set of Lipschitz functions so that if f is Lipschitz, df is a well-defined element of $\mathcal{E}(\mu)$ and $\|df\|_{\mathcal{E}(\mu)} \leq \mathbf{L}(f)$.

Definition 2.19 (Push-forward / Pull-back)**.** Let $F : X \to Y$ be Lipschitz and μ a Radon measure on X such that $F_\#\mu$ is a Radon measure on Y. The **push-forward** map

$$(2.20) \qquad F_\# : \mathcal{X}(\mu) \to \mathcal{X}(F_\#\mu)$$

associates to $D \in \mathcal{X}(\mu)$ the unique $F_\# D \in \mathcal{X}(F_\#\mu)$ such that:

$$(2.21) \quad \int_X g \circ F \, D(f \circ F) \, d\mu = \int_Y g \, (F_\# D) f \, dF_\#\mu$$

$$(\forall (g, f) \in C_c(Y) \times \mathrm{Lip}_\mathrm{b}(Y)).$$

The dual map of $F_\#$ is the **pull-back**:

$$(2.22) \qquad \begin{aligned} F^\# : \mathcal{E}(F_\#\mu) &\to \mathcal{E}(\mu) \\ df &\mapsto d(f \circ F). \end{aligned}$$

Definition 2.23 (Exterior powers)**.** Similarly as for vector fields and differential forms, it is possible to define the exterior powers $\mathcal{X}^k(\mu)$ and $\mathcal{E}^k(\mu)$ (see Definition 7.9 and Remark 5.1 in [Sch14a]). We also let $\mathcal{X}^0(\mu) = \mathcal{E}^0(\mu) = L^\infty(\mu)$. Properties of $\mathcal{X}^k(\mu)$ and $\mathcal{E}^k(\mu)$ that we are going to use are:

(Ex1): $\mathfrak{X}^k(\mu)$ and $\mathcal{E}^k(\mu)$ are $L^\infty(\mu)$-normed modules and finite linear combinations of k-fold exterior powers (the simple vectors in the classical algebraic sense) such as

$$(2.24) \qquad D_1 \wedge \cdots \wedge D_k, \quad df_1 \wedge \cdots \wedge df_k,$$

are dense;

(Ex2): If $\mathfrak{X}(\mu)$ or $\mathcal{E}(\mu)$ are finitely generated, the k-fold exterior products of the generators provide a generating set;

(Ex3): There are exterior products

$$(2.25) \qquad \begin{aligned} \wedge &: \mathfrak{X}^k(\mu) \times \mathfrak{X}^l(\mu) \to \mathfrak{X}^{k+l}(\mu) \\ \wedge &: \mathcal{E}^k(\mu) \times \mathcal{E}^l(\mu) \to \mathcal{E}^{k+l}(\mu) \end{aligned}$$

which are bilinear and have norm at most 1;

(Ex4): There is a natural bilinear pairing:

$$(2.26) \qquad \langle \cdot, \cdot \rangle : \mathfrak{X}^k(\mu) \times \mathcal{E}^k(\mu) \to L^\infty(\mu)$$

which satisfies:

$$(2.27)$$
$$\langle D_1 \wedge \cdots \wedge D_k, df_1 \wedge \cdots \wedge df_k \rangle = \det(D_i f_j)_{i,j}$$
$$\left| \langle \vec{\xi}, \omega \rangle \right| \leq k! |\vec{\xi}|_{\mathfrak{X}^k(\mu)} |\omega|_{\mathcal{E}^k(\mu)} \quad (\forall (\vec{\xi}, \omega) \in \mathfrak{X}^k(\mu) \times \mathcal{E}^k(\mu));$$

(Ex5): Given $\vec{\xi} \in \mathfrak{X}^k(\mu)$, $\omega \in \mathcal{E}^m(\mu)$ for $m \leq k$ the **interior product** $\vec{\xi} \llcorner \omega \in \mathfrak{X}^{k-m}(\mu)$ is defined so that for each $\tilde{\omega} \in \mathcal{E}^{k-m}(\mu)$ one has:

$$(2.28) \qquad \langle \vec{\xi} \llcorner \omega, \tilde{\omega} \rangle = \langle \vec{\xi}, \omega \wedge \tilde{\omega} \rangle.$$

2.2. Metric currents.

Definition 2.29 (Metric currents). A k**-dimensional metric current** T in X is a pair $(\mu, \vec{T})$ where μ is a Radon measure on X and $\vec{T} \in \mathfrak{X}^k(\mu)$. Given $\omega \in \mathcal{E}^k(\mu)$ with integrable local norm, i.e. $|\omega|_{\mathcal{E}^k(\mu)} \in L^1(\mu)$, we let:

$$(2.30) \qquad T(\omega) = \int_X \langle \vec{T}, \omega \rangle \, d\mu.$$

If $f_0, \cdots, f_k$ are Lipschitz functions such that

$$(2.31) \qquad \omega = f_0 df_1 \wedge \cdots \wedge df_k$$

has μ-integrable local norm, we just let:

$$(2.32) \qquad T(f_0, f_1, \cdots, f_k) = T(f_0 df_1 \wedge \cdots \wedge df_k).$$

The **mass measure** of T is $\|T\| = |\vec{T}|_{\mathfrak{X}^k(\mu)} \mu$, and if this measure is finite, then T **has finite mass**; in this case the **mass-norm** of T is:

$$(2.33) \qquad \mathbf{M}(T) = \int_X |\vec{T}|_{\mathfrak{X}^k(\mu)} \, d\mu.$$

The **support** $\operatorname{spt} T$ of T is the support of $\|T\|$.

Definition 2.34 (Boundary and normality). Let T be a metric current and $\{f_i\}_{i=0}^k \subset \operatorname{Lip}_b(X)$; we define:

$$(2.35) \qquad \partial T(f_0, f_1, \cdots, f_{k-1}) = T(df_0 \wedge df_1 \wedge \cdots \wedge df_{k-1});$$

if there is a Radon measure ν such that whenever

$$(2.36) \qquad |df_0 \wedge df_1 \wedge \cdots \wedge df_{k-1}|_{\mathcal{E}^k(\mu)} \in L^1(\mu)$$

one has:

$$(2.37) \qquad |\partial T(f_0, f_1, \cdots, f_{k-1})| \leq \prod_{i=1}^{k-1} \mathbf{L}(f_i) \int_X |f_0|\, d\nu,$$

then **the boundary of** ∂T of T is still a metric current, i.e. one can find $\vec{\partial T} \in \mathfrak{X}^{k-1}(\nu)$ such that:

$$(2.38) \qquad \partial T(f_0, f_1, \cdots, f_{k-1}) = \int_X f_0 \langle \vec{\partial T}, df_1 \wedge \cdots \wedge df_{k-1} \rangle\, d\nu.$$

A metric current T whose boundary is still a current is called **normal**; if both $\|T\|$ and $\|\partial T\|$ are finite measures we let the **normal mass-norm** be:

$$(2.39) \qquad \mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T).$$

A normal current T such that $\|\partial T\|$ is locally finite satisfies the following **joint continuity axiom** (Theorem 5.35 in [Sch14a]): if $\{f_{i,n}\}_{i=0,\cdots,k;n\in\mathbb{N}\cup\{\infty\}}$ satisfy:

$$(2.40) \qquad f_{i,n} \xrightarrow{\text{w}^*} f_{i,\infty} \quad (\forall i, \text{ as } n \to \infty)$$

and if the measures

$$(2.41) \qquad |f_{0,n} df_{1,n} \wedge \cdots \wedge df_{k,n}|_{\mathfrak{X}^k(\|T\|)} \|T\| \quad (n \in \mathbb{N} \cup \{\infty\})$$

are tight, then:

$$(2.42) \qquad \lim_{n\to\infty} T(f_{0,n}, f_{1,n}, \cdots, f_{k,n}) = T(f_{0,\infty}, f_{1,\infty}, \cdots, f_{k,\infty}).$$

Definition 2.43 (Push-forward, interior product). Let $F : X \to Y$ be Lipschitz and μ a Radon measure on X such that $F_\#\mu$ is a Radon measure on Y. The map $F_\# : \mathfrak{X}(\mu) \to \mathfrak{X}(F_\#\mu)$ induces maps

$$(2.44) \qquad \begin{aligned} F_\# &: \mathfrak{X}^k(\mu) \to \mathfrak{X}^k(F_\#\mu) \\ F^\# &: \mathcal{E}^k(F_\#\mu) \to \mathcal{E}^k(\mu). \end{aligned}$$

If $T = (\mu, \vec{T})$ we let $F_\# T$ denote the **push-forward**:

$$(2.45) \qquad F_\# T = (F_\#\mu, F_\#\vec{T}).$$

If $\omega \in \mathcal{E}^m(\mu)$ we let:

$$(2.46) \qquad T \llcorner \omega = (\mu, \vec{T} \llcorner \omega).$$

Definition 2.47 (Weak topology). We say that a sequence of k-dimensional currents $\{T_n\}$ converges to a k-dimensional current T in the **weak topology** if whenever $\{f_i\}_{i=0}^k$ are Lipschitz functions such that for each i, n spt $f_i \cap$ spt T_n is compact, one has:

$$(2.48) \qquad \lim_{n\to\infty} T_n(f_0, f_1, \cdots, f_k) = T(f_0, f_1, \cdots, f_k).$$

The **flat norm** of a current T is:

$$(2.49) \qquad \text{Flat}(T) = \inf\{\mathbf{M}(S_1) + \mathbf{M}(S_2) : T = S_1 + \partial S_2 \text{ for } S_1, S_2 \text{ currents}\}.$$

In particular, if $\text{Flat}(T_n) \to 0$, then $T_n \to 0$ in the weak topology.

2.3. Nagata dimension.

Definition 2.50 (Nagata cover). Let $(C, s, N) \in (0, \infty)^2 \times (\mathbb{N} \cup \{0\})$; a (C, s, N)-**Nagata cover** of X is a collection of $(N+1)$-families of sets $\{\mathcal{C}_i\}_{i=0,\cdots,N}$ such that:

(**NSep**): If $A, B \in \mathcal{C}_i$ are distinct then:

$$(2.51) \qquad \qquad \mathrm{dist}(A, B) \geq s;$$

(**NBd**): For each $A \in \mathcal{C}_i$:

$$(2.52) \qquad \qquad \mathrm{diam}\, A \leq Cs.$$

A Nagata cover is **sorted** if $i \geq 1$ and $B \in \mathcal{C}_i$ imply that for some $A \in \mathcal{C}_{i-1}$ one has

$$(2.53) \qquad \qquad \mathrm{dist}(A, B) < s.$$

Note that from any Nagata cover one can produce a sorted one by moving sets across the families $\{\mathcal{C}_i\}_{i=0,\cdots,N}$.

Definition 2.54 (Nagata dimension). A metric space X has **Nagata dimension** at most N if there is a $C > 0$ (cover-separation parameter) such that for each $s > 0$ X admits a (C, s, N)-Nagata cover. The Nagata dimension of X is the smallest N so that X has Nagata dimension at most N.

A metric space X has **small Nagata dimension** at most N if for each $x \in X$ there is an $r > 0$ (scale parameter) such that $B(x, r)$ has Nagata dimension at most N. Note that it might happen that the scale r might be chosen uniformly, i.e. independently of x. The small Nagata dimension of X is the smallest N so that X has small Nagata dimension at most N.

Theorem 2.55 (Polyhedral approximation). *X has Nagata dimension at most N if and only if there is a constant C_{p} depending only on the parameters (C, N) in the definition of Nagata dimension, such that for each $s > 0$ there is an N-dimensional simplicial complex P, equipped with a metric d_P which restricts on each simplex to a metric induced by a Eucidean norm, and a C_{p}-Lipschitz map*

$$(2.56) \qquad \qquad F : X \to P$$

such that

$$(2.57) \qquad \qquad \|F^* d_P - d_X\|_\infty \leq C_{\mathrm{p}} s.$$

Proof. Sufficiency follows from [LS05, Prop. 2.5]; we focus on necessity.
Step 1: Construction of P and F.
Take a sorted (C, s, N)-Nagata cover $\{\mathcal{C}_i\}_{i=0,\cdots,N}$; to each $S_i \in \mathcal{C}_i$ associate a $(3s^{-1})$-Lipschitz function:

$$(2.58) \qquad \qquad \Phi_{S_i} : X \to [0, 1]$$

such that:

$$(2.59) \qquad \qquad \Phi_{S_i} = \begin{cases} 0 & \text{on } X \setminus B(S_i, s/3) \\ 1 & \text{on } S_i. \end{cases}$$

By **(NSep)** for each $x \in X$ ane each i there is at most one $S_i \in \mathcal{C}_i$ such that $\Phi_{S_i}(x) \neq 0$ and so

$$(2.60) \qquad 1 \leq \sum_{i=0}^{N} \sum_{S_i \in \mathcal{C}_i} \Phi_{S_i} \leq N+1,$$

and we can rescale the Φ_{S_i} by their sum to get a $(C(N)s^{-1})$-Lipschitz partition of unity (still denoted by $\{\Phi_{S_i}\}$):

$$(2.61) \qquad \sum_{i=0}^{N} \sum_{S_i \in \mathcal{C}_i} \Phi_{S_i}(x) = 1.$$

Let $\mathcal{S}$ denote the metric space whose points are the $\{S_i \in \mathcal{C}_i\}_{i=0,\cdots,N}$ and whose distance is the Hausdorff distance. We embedd $\mathcal{S}$ in l^{∞} and let $[S_i]$ denote the image of S_i. We define

$$F : X \to l^{\infty}$$
$$(2.62)$$
$$x \mapsto \sum_{i=0}^{N} \sum_{S_i \in \mathcal{C}_i} \Phi_{S_i}(x)[S_i],$$

and note that $F(X) \subset P$, where P is the N-dimensional simplicial complex obtained by taking the convex hull of all finite tuples $([S_{i_0}], \cdots, [S_{i_k}])$ whenever

$$(2.63) \qquad \Phi_{S_{i_0}} \cdot \Phi_{S_{i_1}} \cdots \Phi_{S_{i_k}} \neq 0.$$

Note that the metric d_P is the restriction of the ambient metric of l^{∞}.
`Step 2: Proof of (2.57).`
For $x \in X$ we let $L(x) = \{S_i : \Phi_{S_i}(x) \neq 0\}$. If $S_x, S \in L(x)$ then:

$$(2.64) \qquad \operatorname{dist}(S, S_x) \leq \frac{2}{3}s,$$

and so we have the bound on the Hausdorff-distance:

$$(2.65) \qquad d_H(S, S_x) \leq (\frac{2}{3} + C)s.$$

If $S_x \in L(x)$, $S_y \in L(y)$ we then have:

$$(2.66) \qquad \|F(x) - [S_x]\|_{l^{\infty}} \leq (\frac{2}{3} + C)s$$

$$(2.67) \qquad \|F(y) - [S_y]\|_{l^{\infty}} \leq (\frac{2}{3} + C)s.$$

As $d(x, S_x) \leq s/3$, $d(y, S_y) \leq s/3$ we conclude that

$$(2.68) \qquad |d(x,y) - d_H(S_x, S_y)| \leq 2(\frac{1}{3} + C)s,$$

from which we get:

$$(2.69) \qquad |\|F(x) - F(y)\|_{l^{\infty}} - d(x,y)| \leq (2 + 4C)s.$$

`Step 3: Uniform Lipschitz bound on `F`.`
If $L(x) \cap L(y) = \emptyset$ then $d(x,y) \gtrsim s$ and so a uniform bound on:

$$(2.70) \qquad \frac{\|F(x) - F(y)\|_{l^{\infty}}}{d(x,y)}$$

follows from (2.57).

Choose $W_{x,y} \in L(x) \cap L(y)$; then:

(2.71)
$$
\begin{aligned}
F(x) - F(y) &= \sum_{S \in L(x)} \Phi_S(x)[S] - \sum_{T \in L(y)} \Phi_T(y)[T] \\
&= \sum_{S \in L(x)} \Phi_S(x)[S] - \sum_{S \in L(x)} \Phi_S(x)[W_{x,y}] \\
&\quad + \sum_{T \in L(y)} \Phi_T(y)[W_{x,y}] - \sum_{T \in L(y)} \Phi_T(y)[T] \\
&= \sum_{S \in L(x) \setminus \{W_{x,y}\}} \Phi_S(x)([S] - [W_{x,y}]) - \sum_{T \in L(y) \setminus \{W_{x,y}\}} \Phi_T(y)([T] - [W_{x,y}]) \\
&= \sum_{S \in L(x) \setminus L(y)} (\Phi_S(x) - \Phi_S(y))([S] - [W_{x,y}]) - \sum_{T \in L(y) \setminus L(x)} (\Phi_T(y) - \Phi_T(x))([T] - [W_{x,y}]) \\
&\quad + \sum_{W \in L(x) \cap L(y)} (\Phi_W(x) - \Phi_W(y))([W] - [W_{x,y}]).
\end{aligned}
$$

Now if $Z \in L(x) \cup L(y)$ and $W \in L(x) \cap L(y)$ we have:

(2.72)
$$
\|[Z] - [W_{x,y}]\|_{l^\infty} \le \left(\frac{2}{3} + C\right)s
$$
$$
\|[W] - [W_{x,y}]\|_{l^\infty} \le \left(\frac{2}{3} + C\right)s.
$$

Then by (2.71):

(2.73)
$$
\begin{aligned}
\|F(x) - F(y)\|_{l^\infty} &\le \frac{3}{s} d(x,y) \left(\#(L(x) \setminus L(y)) + \#(L(y) \setminus L(x)) + \#(L(x) \cap L(y))\right) \left(\frac{2}{3} + C\right)s \\
&\le 6N\left(\frac{2}{3} + C\right)d(x,y).
\end{aligned}
$$

$\square$

Definition 2.74 ((TAP)(N)). A metric space has the property **tower of approximations by N-dimensional polyhedra** (abbr. **(TAP)**(N)) if there are constants C, $\{C_n\}$, $s_n \searrow 0$, and N-dimensional polyhedral complexes P_n (where the metric restricts on each simplex to a metric induced by a norm), and C-Lipschitz maps:

(2.75)
$$
F_n : X \to P_n
$$

such that:

(2.76)
$$
\|F_n^* d_{P_n} - d_X\|_\infty \le C s_n
$$
(2.77)
$$
F_n^* d_{P_n} \le C_n F_{n+1}^* d_{P_{n+1}}.
$$

Equivalently, (2.77) can be reformulated by asking for C_n-Lipschitz maps π_n : $P_{n+1} \to P_n$ which make the following diagram commute:

(2.78)

$$
\begin{array}{ccc}
 & F_n \nearrow & P_n \\
X & & \uparrow \pi_n \\
 & F_{n+1} \searrow & P_{n+1}
\end{array}
$$

Theorem 2.79 (finite-dimensionality from **(TAP)**(N)). *Let X have property **(TAP)**(N) and μ be a Radon measure on X. Then the analytic dimension of (X, μ) is at most N.*

Proof. `Step 1:  Weak* approximation by Lipschitz maps.`
   Let $f : X \to \mathbb{R}$ be 1-Lipschitz. Then there is a function $\tilde{f} : P_n \to \mathbb{R}$ with Lipschitz constant $\mathbf{L}(\tilde{f}) \le L(C)$ such that:

(2.80)
$$\|f - \tilde{f} \circ F_n\|_\infty \le L(C) s_n.$$

In fact, it suffices to select a maximal $4C s_n$-separated net $\mathcal{S}$ in X, let $\tilde{f}(F_n(x)) = f(x)$ for $x \in \mathcal{S}$ and extend $\tilde{f}$ by MacShane's Lemma.
`Step 2:  Mazur's Lemma`
   Assume that μ is a finite Borel measure on X and

(2.81)
$$\{D_1, \cdots, D_k\} \subset \mathfrak{X}(\mu)$$

are independent so that there are 1-Lipschitz functions

(2.82)
$$\{g_1, \cdots, g_k\}$$

such that the matrix $(D_i g_j)$ has μ-a.e. rank k. By `Step 1` each g_j can be approximated in the weak* topology by a sequence $g_j^{(n)} \circ F_n$ where $g_j^{(n)} : P_n \to \mathbb{R}$.

   As $D_i(g_j^{(n)} \circ F_n) \xrightarrow{\text{w}^*} D_i g_j$, using Mazur's Lemma, we can find finite convex linear combinations $\sum_n t_n^{(m)} g_j^{(n)}$ such that:

(2.83)
$$D_i \left(\sum_n t_n^{(m)} g_j^{(n)} \circ F_n \right) \xrightarrow{L^2(\mu)} D_i g_j,$$

where note that $t_n^{(m)}$ does not depend on i.
   In particular, for m sufficiently large we can find a set $K \subset X$ of positive measure on which:

(2.84)
$$\left| \det \left(D_i \left(\sum_n t_n^{(m)} g_j^{(n)} \circ F_n \right) \right)_{i,j} \right| > 0;$$

as the sum $\sum_n t_n^{(m)} g_j^{(n)}$ is finite, and as by **(TAP)**(N) any $\psi : P_n \to \mathbb{R}$ can be written as $\psi = \tilde{\psi} \circ \pi_n$ where $\pi_n : P_{n+1} \to P_n$ is C_n-Lipschitz, we can find n and Lipschitz functions $\tilde{g}_j : P_n \to \mathbb{R}$ such that:

(2.85)
$$|\det(D_i(\tilde{g}_j \circ F_n))_{i,j}| > 0,$$

on a subset $\tilde{K}$ of positive measure.

If $\tilde{\mu}$ denotes the disintegration of $\mu \llcorner \tilde{K}$ with respect to $F_{n\#}\mu \llcorner \tilde{K}$, then for $F_{n\#}\mu \llcorner \tilde{K}$-a.e. p $D_i(g_j \circ F_n)$ is $\tilde{\mu}(p)$-a.e. constant on $F_n^{-1}(p)$. Thus the $d\tilde{g}_j$ are independent in $\mathcal{E}(F_{n\#}\mu \llcorner \tilde{K})$; as $F_{n\#}\mu \llcorner \tilde{K}$ is a Radon measure supported in an N-dimensional simplicial complex, we must have $k \leq N$ by Example 2.13. $\qquad \square$

3. Inverse Systems

In this Section we first discuss inverse systems of cube complexes which yield Poincaré inequalities. Definition 3.1 and Theorem 3.8 are contained in [CK15, Sec. 11]; note that in general the constant in the Poincaré inequality might depend both on the location of the ball B in the space and its radius $\mathrm{rad}(B)$, because the geometry at ∞ of such cube complexes might be complicated. We then consider inverse systems which satisfy a relaxed set of axioms, Definition 3.14, which is suitable for constructing normal currents (Defn. 3.16). The description of how to produce such currents and the kind of "calculus" supported by such spaces is discussed in Theorem 3.20.

For a cube complex X we will let $\mathrm{Cell}_k(X)$ denote the set of its k-dimensional cells.

Definition 3.1 (Admissible inverse systems / AIS). Let $(N, m) \in \mathbb{N} \times (\mathbb{N} \cap [2, \infty))$ and consider a collection of metric measure spaces $\{(X_i, \mu_i)\}_{i \in I}$ and maps $\{\pi_i\}_{i \in I}$ where the index set I is of the form $\{k \in \mathbb{Z} : k \geq k_0\}$. This collection $\{(X_i, \mu_i)\}_{i \in I}$ is an **(N-dimensional) admissible inverse system** if the following axioms hold.

(**IBGeom**): Each X_i is a nonempty connected cube-complex (with the length metric) which is a union of its N-dimensional cells which are isometric to the Euclidean cube $[0, m^{-i}]^N$; moreover, there is a uniform bound C_{geo} on the cardinality of each link.

Let $X_i^{(1)}$ denote the cube-complex obtained from X_i by subdividing each N-cube of X_i into m^N isometric subcubes; when the subdivision operation is repeated k-times we use the notation $X_i^{(k)}$.

(**IOpen**): Each map $\pi_i : X_{i+1} \to X_i^{(1)}$ is open, surjective, cellular, and restricts to an isometry on every face.

A **gallery** in $X_i^{(k)}$ is a finite sequence of N-dimensional cells

$$(3.2) \qquad \{\sigma_1, \cdots, \sigma_l\} \subset \mathrm{Cell}_N(X_i^{(k)})$$

such that each σ_k and σ_{k+1} share an $(N-1)$-dimensional face. If $y \in \sigma_1$ and $y' \in \sigma_l$ we say that the gallery connects y to y'.

(**IGall**): Any two points in X_i are connected by a gallery. For each $x \in X_i^{(1)}$, and for each $y, y' \in \pi_i^{-1}(x)$ there is a gallery (in X_{i+1}) of at most C_{gall}-cells joining y to to y';

(**IMeas**): Each μ_i restricts to a constant multiple (with weight $\mathrm{weight}(\mu_i, \sigma)$) of Lebesgue measure on each element σ of $\mathrm{Cell}_N(X_i)$ and $\pi_{i\#}\mu_{i+1} = \mu_i$. Moreover, there is a uniform constant (in i) C_μ such that whenever $\sigma, \sigma' \in \mathrm{Cell}_N(X_i)$ are adjacent:

$$(3.3) \qquad \mu_i(\sigma') \leq C_\mu \mu_i(\sigma);$$

(IPoinc): Let $f_i \in \mathrm{Cell}_{N-1}(X_i^{(1)})$ and $f_{i+1} \in \pi_i^{-1}(f_i)$; then the quantity:

$$(3.4) \qquad \sum_{\tau \in \mathrm{Bd}(f_{i+1}) \in \pi_i^{-1}(\sigma_i)} \frac{\mathrm{weight}(\mu_{i+1}, \tau)}{\mathrm{weight}(\mu_i, \sigma_i)}$$

is constant as σ_i varies on the set $\mathrm{Bd}(f_i)$ of N-cells of $X_i^{(1)}$ which bound f_i.

For a discussion of the Poincaré inequality we refer the reader to[HK98, Kei03].

Definition 3.5 (Local PI-space). A geodesic metric measure space (X, μ) is a **local $(1, p)$-PI space** if μ is locally doubling, i.e. for each ball $B \subset X$ there is a constant $C_\mathrm{d}(B)$ such that $\mu \llcorner B$ is $C_\mathrm{d}(B)$-doubling as a measure on the metric space B, and if there is a constant $C_\mathrm{PI}(B)$ such that for each ball $B' \subset B$ and each Lipschitz function $f : X \to \mathbb{R}$ one has:

$$(3.6) \qquad \fint_{B'} |f - f_{B'}| \, d\mu \le C_\mathrm{PI}(B) \mathrm{rad}(B') \left(\fint_{B'} (\mathrm{Lip} f)^p \, d\mu \right)^{1/p},$$

where

$$(3.7) \qquad \mathrm{Lip} f(x) = \limsup_{y \to x, y \ne x} \frac{|f(y) - f(x)|}{d(x, y)};$$

if $C_\mathrm{PI}(B)$ and $C_\mathrm{d}(B)$ can be chosen independent of B then (X, μ) is a called a $(1, p)$-PI space.

This Theorem summarizes properties of AISs.

Theorem 3.8 (Inverse limits are local $(1, 1)$-PI). *Let $\{(X_i, \mu_i)\}_{i \in I}$ be an admissible inverse system and let $\{p_i\}_{i \in I} \subset \prod_{i \in I} X_i$ be a compatible collection of basepoints, i.e. $\pi_i(p_{i+1}) = p_i \ \forall i$. Then the following limit (in the pointed measured Gromov-Hausdorff sense) exists*

$$(3.9) \qquad \lim_{k \to \infty} (X_k, \mu_k, p_k) = (X_\infty, \mu_\infty, p_\infty)$$

*and is called the **inverse** limit of $\{(X_i, \mu_i)\}_{i \in I}$ (given the choice of basepoints). Then the inverse limit is a local $(1, 1)$-PI space where $C_\mathrm{PI}(B)$ and $C_\mathrm{d}(B)$ depend, besides B, only on C_geo, m, C_gall and C_μ. If some (X_k, μ_k, p_k) is a $(1, 1)$-PI space, so is the inverse limit.*

Proof. Step 1: Existence of the inverse limit.

By **(IBGeom)**, **(IOpen)** and **(IGall)** there is a constant $C = C(C_\mathrm{geo}, C_\mathrm{gall}, m)$ such that:

$$(3.10) \qquad \|\pi_i^* d_{X_{i+1}} - d_{X_i}\| \le C m^{-i}.$$

By **(IMeas)** for each $R > 0$ there is a constant $C = C(R, m, C_\mu)$ such that, if $f : X_i \to [0, 1]$ is 1-Lipschitz with $\mathrm{spt} f \subset B(p_i, R)$ then:

$$(3.11) \qquad \left| \int_{X_i} f \, d\mu_i - \int_{X_i} f \, d(\pi_{i,\#} \mu_{i+1}) \right| \le C m^{-i}.$$

Choosing an appropriate metric to metrize the mGH-topology we conclude that the sequence $\{(X_k, \mu_k, p_k)\}_{k \ge \inf I}$ is Cauchy.

Step 2: The Poincaré inequality.

Let $k_0 \in I$ and $B \subset X_\infty$ be a ball. As the induced map:

$$(3.12) \qquad \pi_{\infty,k_0} : X_\infty \to X_{k_0} \quad (\lim_{i \to \infty} \text{ of } \pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_{k_0})$$

is 1-Lipschitz, there is a finite subcomplex $S_B \subset X_{k_0}$ whose interior contains $\pi_{\infty,k_0}(B)$; without loss of generality we may assume S_B gallery-connected; then S_B is a $(1,1)$-PI space since pairs of points in S_B can be joined by pencils of curves that satisfy an appropriate modulus estimate [Kei03].

The argument in [CK15, Sec. 11] shows that, if $(C_{S_B}, C_\mathrm{d}(S_B))$ are the constants in the Poincaré inequality and the doubling condition for $(S_B, \mu_{k_0} \llcorner S_B)$, then $(\pi_{\infty,k_0}^{-1}(S_B), \mu_\infty \llcorner \pi_{\infty,k_0}^{-1}(S_B))$ is a $(1,1)$-PI space with constant in the Poincaré inequality:

$$(3.13) \qquad C = C(C_{S_B}, C_\mathrm{d}(S_B), m, C_\mathrm{gall}, C_\mu, C_\mathrm{geo}).$$

The fact that in (3.6) one can take the same ball on both sides follows from [HK95] because by **(IGall)** the interior of S_B satisfies an appropriate *chain condition*, which passes to $\pi_{\infty,k_0}^{-1}(S_B)$ because of **(IOpen)**.

Finally note that if (X_{k_0}, μ_{k_0}) is a $(1,1)$-PI space the constants C_{S_B}, $C_\mathrm{d}(S_B)$ can be assumed independent of S_B and so (X_∞, μ_∞) is a $(1,1)$-PI space. $\qquad \square$

The following definition introduces the kind of systems that we use to build normal currents.

Definition 3.14 (Weak Admissible Inverse Systems / WAIS). Let $\{(X_i, \mu_i)\}_{i \in I}$ satisfy **(IBGeom)**–**(IMeas)**. We say that $\{(X_i, \mu_i)\}_{i \in I}$ is a **weak admissible inverse system** if the following axioms hold:

(IOr): Each $\sigma \in \mathrm{Cell}_N(X_i)$ carries an orientation, and these orientations induce compatible orientations on the cells in $\mathrm{Cell}_N(X_i^{(k)})$ for each $k \geq 1$. These orientations are compatible in the sense that if $\sigma \in \mathrm{Cell}_N(X_{i+1})$ then the orientation of $\pi_i(\sigma)$ is induced by π_i;

(IFlux): There is a $k_0 \in I$ such that for $i \geq k_0$ the following holds. Fix $f_i \in \mathrm{Cell}_{N-1}(X_i^{(1)})$ in the interior of some cell in $\mathrm{Cell}_N(X_i)$ and partition $\mathrm{Bd}(f_i)$ in two subsets $\mathrm{Bd}(f_i, +)$ and $\mathrm{Bd}(f_i, -)$ depending on which orientation they induce on f_i. Then for each $f_i \in \mathrm{Cell}_{N-1}(X_i^{(1)})$ and each $f_{i+1} \in \pi_i^{-1}(f_i)$ the following holds:

(3.15)
$$\sum_{\tau \in \pi_i^{-1}(\mathrm{Bd}(f_i,+)) \cap \mathrm{Bd}(f_{i+1}, X_i^{(1)})} \mathrm{weight}(\mu_{i+1}, \tau) = \sum_{\tau \in \pi_i^{-1}(\mathrm{Bd}(f_i,-)) \cap \mathrm{Bd}(f_{i+1}, X_i^{(1)})} \mathrm{weight}(\mu_{i+1}, \tau).$$

In general **(IFlux)** is weaker than **(IPoinc)**.

Definition 3.16 (Normal currents associated to a WAIS). Let $\{(X_i, \mu_i)\}_{i \in I}$ be a WAIS. We can canonically identify each $\sigma \in \mathrm{Cell}_N(X_i)$ with $[0, m^{-i}]^N$ and associate to it a (classical) N-normal current $[\![\sigma]\!]$ by:

$$(3.17) \qquad [\![\sigma]\!] = \pm \partial_1 \wedge \cdots \wedge \partial_N \, \mathcal{L}^N \llcorner \sigma,$$

where the choice of $\pm$ depends on the choice of orientation on σ. To each X_i we can associate a (metric) normal current N_i (where $\|N_i\|$ and $\|\partial N_i\|$ are locally finite) by:

$$(3.18) \qquad N_i = \sum_{\sigma \in \mathrm{Cell}_N(X_i)} \mathrm{weight}(\mu_i, \sigma)[\![\sigma]\!];$$

(IOr) guarantees that:

$$(3.19) \qquad \pi_{i\#}N_{i+1} = N_i.$$

Theorem 3.20 (Limit normal currents for WAIS). *Let $\{(X_i, \mu_i)\}_{i\in I}$ be a WAIS and let $\{p_i\}_{i\in I} \subset \prod_{i\in I} X_i$ be a compatible collection of basepoints. Then:*

   (mGH): *The following limit exists as in Theorem 3.8:*

$$(3.21) \qquad \lim_{k\to\infty}(X_k, \mu_k, p_k) = (X_\infty, \mu_\infty, p_\infty);$$

   (Nag): *For $i \in I \cup \{\infty\}$ the metric space X_i has small Nagata dimension N with uniform parameters (in i: scale and cover-separation). If some X_{k_0} has Nagata dimension N, so do all the X_i with uniform cover-separation parameter;*

   (Wea): *For $i \in I \cup \{\infty\}$ the module $\mathfrak{X}(\mu_i)$ is free on N-generators $\{D_{i,\alpha}\}_{\alpha\in\{1,\cdots,N\}}$. If $i \in \mathbb{Z}$ then for each $\sigma \in \mathrm{Cell}_N(X_i)$:*

$$(3.22) \qquad \chi_\sigma D_{i,\alpha} = \partial_\alpha,$$

   ∂_α being the "Euclidean derivation" in the α-direction in the cell σ. The derivations are compatible in the sense that whenever $i \geq j$:

$$(3.23) \qquad \pi_{i,j\#}D_{i,\alpha} = D_{j,\alpha};$$

   (Nor): *Assume that convergence in (3.21) takes place (as weak* convergence of measures and standard Hausdorff-Vietoris convergence) in some complete separable metric space Z and that there is a constant C_Z such that if $i, j \in I \cup \{\infty\}$ with $i \geq j$ one has:*

$$(3.24) \qquad \sup_{x_i \in X_i} d_Z(x_i, \pi_{i,j}(x_i)) \leq C_Z m^{-j}.$$

   Then the following limit of currents exists in the weak topology for normal currents:

$$(3.25) \qquad \lim_{k\to\infty} N_k = N_\infty,$$

   and

$$(3.26) \qquad \pi_{i,j\#}N_i = N_j;$$

   furthermore, each current N_i is **simple** *in the sense that:*

$$(3.27) \qquad N_i = D_{i,1} \wedge \cdots \wedge D_{i,N}\, \mu_i.$$

   Finally, the convergence in (3.25) does not entail loss of mass: i.e. for each open $U \Subset X_j$:

$$(3.28) \qquad \begin{aligned} \|N_i\|(\pi_{i,j}^{-1}(U)) &= \|N_j\|(U) \\ \|\partial N_i\|(\pi_{i,j}^{-1}(U)) &= \|\partial N_j\|(U). \end{aligned}$$

Remark 3.29 (Assumption (3.24)). Note that assumption (3.24) is not restrictive as by induction and passing to the limit in (3.10) one has:

$$(3.30) \qquad \|\pi_{i,j}^* d_{X_j} - d_{X_i}\| \leq C m^{-j};$$

one can then find a metric on $Z_i = X_\infty \sqcup X_i$ extending d_{X_∞} and d_{X_i} such that (3.24) holds for the pair of indices (∞, i). Then one glues the spaces Z_i across X_∞ and uses the surjectivity of the maps $\{\pi_{i,j}\}$ to deduce (3.24).

Proof of Theorem 3.20. `Step 1: Proof of` **(mGH)** `and` **(Nag)**.

The existence of the inverse limit follows as in Theorem 3.8. The proof of **(Nag)** follows from Theorem 2.79; in fact, if $\sigma \in \mathrm{Cell}_N(X_{k_0})$ we have that the maps:

$$(3.31) \qquad \pi_{\infty,i} : \pi_{\infty,k_0}^{-1}(\sigma) \to \pi_i(\pi_{\infty,k_0}^{-1}(\sigma))$$

show that $\pi_{\infty,k_0}^{-1}(\sigma)$ has property **(TPA)**(N) and so it has small Nagata dimension at most N. A lower bound on the small Nagata dimension follows from a lower bound on the topological dimension as the map in (3.31) is light and so cannot decrease the topological dimension ([Eng78, Thm. 1.24.4]).

The claim about the cover-separation and scale being uniform holds as all elements of $\sigma \in \mathrm{Cell}_N(X_{k_0})$ are isometric to a rescaled copy of $[0,1]^N$. The claim about the large Nagata dimension follows because in that case one can replace σ by X_{k_0} (and the Nagata covers for X_{k_0} can be also used to produce Nagata covers for X_j if $j < k_0$ at scales $> m^{-j}$).

`Step 2: Definition of maximal galleries.`

We start the proof of **(Wea)** by constructing the derivations $D_{i,\alpha}$; without loss of generality we take $\alpha = 1$ and by locality assume that $\inf I = 0$, $X_0 = [0,1]^N$ and $\mu_0 = \mathcal{L}^N \llcorner [0,1]^N$. As the coordinate functions x_α on $[0,1]^N$ can be canonically pulled back to each X_i we will just write x_α for $x_\alpha \circ \pi_{i,j}$ in the following.

A string of cells:

$$(3.32) \qquad \{\sigma_0, \cdots, \sigma_t\} \subset \mathrm{Cell}_N(X_i^{(k)})$$

is an x_1-**gallery** if:

> **(Mx1):** $\sigma_i \neq \sigma_{i+1}$ and $\max x_1(\sigma_i) = \min x_1(\sigma_{i+1})$ for $0 \leq i \leq t-1$;
> **(Mx2):** σ_i and σ_{i+1} share a codimension-1 face on which x_1 is constant.

If an x_1-gallery g contains an x_1-gallery g' we say that g **extends** g'; if g does not admit an x_1-gallery properly extending it, then g is called **maximal** and the set of maximal x_1-galleries is denoted by $\mathrm{Mx}(X_i^{(k)})$. Given $\sigma \in \mathrm{Cell}_N(X_i^{(k)})$ we use $\partial_+\sigma$ to denote the union of the cells of $\mathrm{Cell}_N(X_i^{(k)})$ which bound the $(N-1)$-dimensional face of σ on which x_1 is maximal and induce on it an orientation opposite to that induced by σ ($\partial_-\sigma$ is defined similarly considering the face on which x_1 is minimal).

To $g \in \mathrm{Mx}(X_i^{(k)})$ we can associate the measure $\mathcal{L}^N \llcorner g$ which is just Lebesgue measure on each cell of g. We now construct measures Q_i on $\mathrm{Mx}(X_i)$ such that:

$$(3.33) \qquad \mu_i = \sum_{g \in \mathrm{Mx}(X_i)} \mathcal{L}^N \llcorner g \, Q_i(g).$$

Note that each $g \in \mathrm{Mx}(X_i^{(k)})$ is contained in a unique $\hat{g} \in \mathrm{Mx}(X_i)$ and if we let $Q_i^{(k)}(g) = Q_i(\hat{g})$, (3.33) implies the more general version:

$$(3.34) \qquad \mu_i = \sum_{g \in \mathrm{Mx}(X_i^{(k)})} \mathcal{L}^N \llcorner g \, Q_i^{(k)}(g).$$

As $\mathrm{Mx}(X_0)$ is a singleton, we let $Q_0 = 1$ on $\mathrm{Mx}(X_0)$ so that (3.33) holds for $i = 0$. The measure Q_{i+1} is defined by recursion: given

$$(3.35) \qquad g = \{\sigma_1, \cdots, \sigma_{m^{i+1}}\} \in \mathrm{Mx}(X_{i+1})$$

we let:

$$(3.36) \quad Q_{i+1}(g) = Q_i^{(1)}(\pi_i(g)) \times \frac{\mu_{i+1}(\sigma_1)}{\mu_i(\pi_i(\sigma_1))} \frac{\mu_{i+1}(\sigma_2)}{\mu_{i+1}(\partial_+\sigma_1)} \times \cdots \times \frac{\mu_{i+1}(\sigma_{m^{i+1}})}{\mu_{i+1}(\partial_+\sigma_{m^{i+1}-1})}.$$

`Step 3: Proof of (3.33).`

We prove (3.33) by induction; assume that it holds for μ_i and let

$$(3.37) \qquad \nu_{i+1} = \sum_{g \in \mathrm{Mx}(X_{i+1})} \mathcal{L}^N \llcorner g \, Q_{i+1}(g);$$

it suffices to show that whenever $\sigma \in \mathrm{Cell}_N(X_{i+1})$ one has:

$$(3.38) \qquad \nu_{i+1}(\sigma) = \mu_{i+1}(\sigma).$$

To σ it is associated a unique $t \in \{1, \cdots, m^{i+1}\}$ such that, whenever $g \in \mathrm{Mx}(X_{i+1})$ extends σ, σ is the t-th element of g. In (3.36) if we sum on $\{\sigma_{t+1}, \cdots, \sigma_{m^{i+1}}\}$ and use **(IFlux)** we get:

(3.39)

$$Q_{i+1}(\{g \text{ extends } \{\sigma_1, \cdots, \sigma_t\}) = Q_i^{(1)}(\{\pi_i(g) \text{ extends } \{\pi_i(\sigma_1), \cdots, \pi_i(\sigma_t)\}))$$
$$\times \frac{\mu_{i+1}(\sigma_1)}{\mu_i(\pi_i(\sigma_1))} \frac{\mu_{i+1}(\sigma_2)}{\mu_{i+1}(\partial_+\sigma_1)} \times \cdots \times \frac{\mu_{i+1}(\sigma_t)}{\mu_{i+1}(\partial_+\sigma_{t-1})}.$$

We will compute $Q_{i+1}(\{g \text{ extends } \{\sigma_t\}\})$ by summing (3.39) on $\sigma_1, \cdots, \sigma_{t-1}$; note that the order of summation matters as σ_t is kept fixed. Concretely, $\sigma_{t-1} \in \partial_-\sigma_t$, $\sigma_{t-2} \in \partial_-\sigma_{t-1}$, etc... We thus start by removing the innermost sums, i.e. start with σ_1, then σ_2, etc... For example, using **(IFlux)** and **(IOpen)**:

(3.40)

$$\sum_{\sigma_1} Q_{i+1}(\{g \text{ extends } \{\sigma_1, \cdots, \sigma_t\}\}) = \sum_{\pi_i(\sigma_1)} Q_i^{(1)}(\{\pi_i(g) \text{ extends } \{\pi_i(\sigma_1), \cdots, \pi_i(\sigma_t)\}\})$$
$$\times \frac{\mu_{i+1}(\sigma_2)}{\mu_i(\pi_i(\sigma_1))} \frac{\mu_{i+1}(\sigma_3)}{\mu_{i+1}(\partial_+\sigma_2)} \times \cdots \times \frac{\mu_{i+1}(\sigma_t)}{\mu_{i+1}(\partial_+\sigma_{t-1})}.$$

But note that for $2 \leq j \leq m$, $\pi_i(\sigma_j)$ and $\pi_i(\sigma_{j-1})$ belong to the same cell of $\mathrm{Cell}_N(X_i)$ implying

$$(3.41) \qquad \mu_i(\pi_i(\sigma_j)) = \mu_i(\pi_i(\sigma_{j-1})),$$

from which we obtain:

(3.42)

$$Q_{i+1}(\{g \text{ extends } \{\sigma_2, \cdots, \sigma_t\}\}) = Q_i^{(1)}(\{\pi_i(g) \text{ extends } \{\pi_i(\sigma_2), \cdots, \pi_i(\sigma_t)\}\})$$
$$\times \frac{\mu_{i+1}(\sigma_2)}{\mu_i(\pi_i(\sigma_2))} \frac{\mu_{i+1}(\sigma_3)}{\mu_{i+1}(\partial_+\sigma_2)} \times \cdots \times \frac{\mu_{i+1}(\sigma_t)}{\mu_{i+1}(\partial_+\sigma_{t-1})}.$$

We can iterate the previous argument up to m; in particular, if $t \leq m$ we would get:

$$(3.43) \qquad Q_{i+1}(\{g \text{ extends } \{\sigma_t\}\}) = \frac{Q_i^{(1)}(\{\pi_i(g) \text{ extends } \{\pi_i(\sigma_t)\}\})}{\mu_i(\pi_i(\sigma_t))} \mu_{i+1}(\sigma_t).$$

We want to generalize (3.43) also for $t > m$ and the main point is illustrated in passing from m to $m+1$. We start with:

(3.44)

$$Q_{i+1}(\{g \text{ extends } \{\sigma_m, \cdots, \sigma_t\}\}) = \frac{Q_i^{(1)}(\{\pi_i(g) \text{ extends } \{\pi_i(\sigma_m), \cdots, \pi_i(\sigma_t)\}\})}{\mu_i(\pi_i(\sigma_m))}$$
$$\times \mu_{i+1}(\sigma_m) \frac{\mu_{i+1}(\sigma_{m+1})}{\mu_{i+1}(\partial_+\sigma_m)} \times \cdots \times \frac{\mu_{i+1}(\sigma_t)}{\mu_{i+1}(\partial_+\sigma_{t-1})}.$$

We now use the definition of Q_i to deduce:

(3.45)

$$Q_i^{(1)}(\{g \text{ extends } \{\tau_m, \cdots, \tau_t\}\}) = \frac{Q_i^{(2)}(\{\pi_{i-1}(g) \text{ extends } \{\pi_{i-1}(\tau_m), \cdots, \pi_{i-1}(\tau_t)\}\})}{\mu_{i-1}(\pi_{i-1}(\tau_m))}$$
$$\times \mu_i(\tau_m) \frac{\mu_i(\tau_{m+1})}{\mu_i(\partial_+ \tau_m)} \times \cdots \times \frac{\mu_i(\tau_t)}{\mu_i(\partial_+ \tau_{t-1})}.$$

Now $\pi_{i-1}(\tau_m)$ and $\pi_{i-1}(\tau_{m+1})$ belong to the same cell of $\text{Cell}_N(X_{i-2})$; we thus have:

(3.46)
$$\mu_i(\pi_{i-1}(\tau_m)) = \mu_i(\pi_{i-1}(\tau_{m+1})),$$

and we can sum (3.45) on τ_m; applying **(IFlux)** and **(IOpen)**, and we thus obtain:

(3.47)

$$\sum_{\tau_m} Q_i^{(1)}(\{g \text{ extends } \{\tau_m, \cdots, \tau_t\}\}) = \frac{Q_i^{(2)}(\{\pi_{i-1}(g) \text{ extends } \{\pi_{i-1}(\tau_{m+1}), \cdots, \pi_{i-1}(\tau_t)\}\})}{\mu_{i-1}(\pi_{i-1}(\tau_{m+1}))}$$
$$\times \mu_i(\tau_{m+1}) \frac{\mu_i(\tau_{m+2})}{\mu_i(\partial_+ \tau_{m+1})} \times \cdots \times \frac{\mu_i(\tau_t)}{\mu_i(\partial_+ \tau_{t-1})}$$
$$= Q_i^{(1)}(\{g \text{ extends } \{\tau_{m+1}, \cdots, \tau_t\}\});$$

thus if in (3.44) we sum over σ_m we obtain:

(3.48)

$$Q_{i+1}(\{g \text{ extends } \{\sigma_{m+1}, \cdots, \sigma_t\}\}) = \frac{Q_i^{(1)}(\{\pi_i(g) \text{ extends } \{\pi_i(\sigma_{m+1}), \cdots, \pi_i(\sigma_t)\}\})}{\mu_i(\pi_i(\sigma_{m+1}))}$$
$$\times \mu_{i+1}(\sigma_{m+1}) \frac{\mu_{i+1}(\sigma_{m+2})}{\mu_{i+1}(\partial_+ \sigma_{m+1})} \times \cdots \times \frac{\mu_{i+1}(\sigma_t)}{\mu_{i+1}(\partial_+ \sigma_{t-1})}.$$

Continuing by induction we establish (3.43).

Now from the inductive hypothesis $\nu_i = \mu_i$ we have:

(3.49)
$$Q_i^{(1)}(\{\pi_i(g) \text{ extends } \pi_i(\sigma_t)\}) = m^{(i+1)N} \mu_i(\pi_i(\sigma_t)),$$

from which (using (3.43)) we get:

(3.50)
$$Q_{i+1}(\{g \text{ extends } \sigma_t\}) = m^{(i+1)N} \mu_{i+1}(\sigma_t)$$

which implies (3.38) for $\sigma = \sigma_t$.

Step 4: Construction of Weaver derivations.

Note that $\pi_{i\#} Q_{i+1} = Q_i^{(1)}$ by the recursive definition of Q_{i+1}. Let $g \in \text{Mx}(X_i^{(k)})$ and use $g(0)$ to denote the first cell of g; then any y on the face f_- of $g(0)$ where x_1 is minimal uniquely determines a unit-speed geodesic segment

(3.51)
$$\gamma : [0,1] \to X_i^{(k)}$$

such that:

(3.52)
$$x_1 \circ \gamma(t) = t$$

(3.53)
$$\text{im } \gamma \subset g;$$

if we choose $y \in f_-$ according to $\mathcal{L}^{N-1} \llcorner f_-$ and use the Fubini representation in the x_1-direction on each cell of g, (3.33) implies the representation

$$(3.54) \qquad \mu_i = \int \mathcal{H}_\gamma^1 \, dP_i^{(k)}(\gamma) \quad (i \in I)$$

where $P_i^{(k)}$ is a probability measure concentrated on the set Ω_i of geodesic segments satisfying (3.51) and (3.52). Note also that cellular subdivision does not affect the probabilities, i.e. $P_i^{(k)} = P_i$ and that $\pi_{i\#} Q_{i+1} = Q_i^{(1)}$ implies

$$(3.55) \qquad \pi_{i\#} P_{i+1} = P_i^{(1)} = P_i.$$

Having arranged mGH-convergence in some (proper and complete) Z we see that the measures $\{P_i\}_{i \in I}$ form a Cauchy sequence in the weak* topology, and passing to a limit we obtain:

$$(3.56) \qquad \mu_\infty = \int \mathcal{H}_\gamma^1 \, dP_\infty(\gamma)$$

where P_∞ is concentrated on Ω_∞. We finally let $D_{i,1}$ be the Weaver derivation associated to the Alberti representation $[P_i, 1]$. To construct $D_{i,\alpha}$ one uses `Step 3` and this step for x_α.

`Step 5: Proof of (Wea).`

Equation (3.22) follows because the Alberti representations used to define $D_{i,\alpha}$ restrict to appropriate Fubini-like representations in the x_α direction on each cell. Equation (3.23) follows from the compatibility (3.55) between the P_i (the case where $i = \infty$ being handled by a limiting argument).

We now let $i \in I \cup \{\infty\}$; we have:

$$(3.57) \qquad D_{i,\alpha} x_\beta = \delta_{\alpha,\beta}$$

so that the index of $\mathcal{X}(\mu_i)$ is at least N. On the other hand by `Step 1` X_i has property **(TAP)**(N) and so $\mathcal{X}(\mu_i)$ must have index N and be free on N generators. If i is finite the fact that $\{D_{i,\alpha}\}_\alpha$ gives a basis of $\mathcal{X}(\mu_i)$ follows immediately from (3.22). To show that $\{D_{\infty,\alpha}\}$ is a basis for $\mathcal{X}(\mu_\infty)$ we choose a basis $\{\tilde{D}_{\infty,\alpha}\}$ for $\mathcal{X}(\mu_\infty)$ satisfying:

$$(3.58) \qquad \left| \tilde{D}_{\infty,\alpha} \right|_{\mathcal{X}(\mu_\infty)} = 1 \quad (\forall \alpha),$$

and then choose a matrix-valued measurable map $(M_{\alpha,\beta})$ with:

$$(3.59) \qquad D_{\infty,\alpha} = \sum_\beta M_{\alpha,\beta} \tilde{D}_{\infty,\beta};$$

using the functions x_γ we arrive at the identity:

$$(3.60) \qquad \mathrm{Id}_N = (D_{\infty,\alpha} x_\gamma) = (M_{\alpha,\beta}) \cdot (\tilde{D}_{\infty,\beta} x_\gamma),$$

which provides a lower bound on $\det(M_{\alpha,\beta})$ showing that $\{D_{\infty,\alpha}\}$ is a basis for $\mathcal{X}(\mu_\infty)$.

`Step 6: Proof of (3.28) when i, j are finite.`

Note that for $i = j + 1$ (3.26) is just (3.19). In particular, as long as i is finite, (3.27) follows from the definition of N_i, (3.18), and the construction of the Weaver derivations $\{D_{i,\alpha}\}$ as long as one chooses the "order" of the coordinates x_α on each cell of $X_{\inf I}$ compatibly with the orientations. Thus, we have:

$$(3.61) \qquad \|N_i\| = \mu_i,$$

and so the first equation in (3.28) follows.

Taking the boundary in (3.18) we get:

(3.62)
$$\partial N_i = \sum_{\sigma \in \mathrm{Cell}_N(X_i)} \mathrm{weight}(\mu_i, \sigma)\, \partial[\![\sigma]\!] = \sum_{\tau \in \mathrm{Cell}_N(X_{i-1}^{(1)})} \sum_{\sigma \in \pi_{i-1}^{-1}(\tau)} \mathrm{weight}(\mu_i, \sigma)\, \partial[\![\sigma]\!]$$

$$= \sum_{f_{i-1} \in \mathrm{Cell}_N(X_{i-1}^{(1)})} \sum_{f_i \in \pi_{i-1}^{-1}(f_{i-1})} \left\{ \sum_{\sigma \in \pi_{i-1}^{-1}(\mathrm{Bd}(f_{i-1},+)) \cap \mathrm{Bd}(f_i, X_i)} \mathrm{weight}(\mu_i, \sigma) \right.$$

$$\left. - \sum_{\sigma \in \pi_{i-1}^{-1}(\mathrm{Bd}(f_{i-1},-)) \cap \mathrm{Bd}(f_i, X_i)} \mathrm{weight}(\mu_i, \sigma) \right\} [\![f_i]\!].$$

Note that in (3.62) the terms which correspond to f_{i-1} lying in the interior of some $\tau \in \mathrm{Cell}_N(X_{i-1})$ vanish because of **(IFlux)**. Let $U \Subset X_{i-1}$ open; then

(3.63)
$$\|\partial N_{i-1}\|(U) = \sum_{f_{i-1} \in \mathrm{Cell}_{N-1}(X_{i-1})} \left| \sum_{\tau \in \mathrm{Bd}(f_{i-1},+)} \mathrm{weight}(\mu_{i-1}, \tau) - \sum_{\tau \in \mathrm{Bd}(f_{i-1},-)} \mathrm{weight}(\mu_{i-1}, \tau) \right|$$

$$\times\ \mathcal{H}^{N-1}(f_{i-1} \cap U);$$

let $\widetilde{\mathrm{Cell}}_{N-1}(X_{i-1}^{(1)})$ denote the set of those $f_{i-1} \in \mathrm{Cell}_{N-1}(X_{i-1}^{(1)})$ on the boundary of some cell of $\mathrm{Cell}_N(X_{i-1})$; then using **(IMeas)**, **(IOr)** and (3.62) we obtain:

(3.64)
$$\|\partial N_{i-1}\|(U) = \sum_{f_{i-1} \in \widetilde{\mathrm{Cell}}_{N-1}(X_{i-1}^{(1)})} \left| \sum_{\tau \in \mathrm{Bd}(f_{i-1},+)} \mathrm{weight}(\mu_{i-1}, \tau) - \sum_{\tau \in \mathrm{Bd}(f_{i-1},-)} \mathrm{weight}(\mu_{i-1}, \tau) \right|$$

$$\times\ \mathcal{H}^{N-1}(f_{i-1} \cap U)$$

$$= \sum_{f_{i-1} \in \widetilde{\mathrm{Cell}}_{N-1}(X_{i-1}^{(1)})} \sum_{f_i \in \pi_{i-1}^{-1}(f_{i-1})} \left| \sum_{\sigma \in \pi_{i-1}^{-1}(\mathrm{Bd}(f_{i-1},+)) \cap \mathrm{Bd}(f_i, X_i)} \mathrm{weight}(\mu_i, \sigma) \right.$$

$$\left. - \sum_{\sigma \in \pi_{i-1}^{-1}(\mathrm{Bd}(f_{i-1},-)) \cap \mathrm{Bd}(f_i, X_i)} \mathrm{weight}(\mu_i, \sigma) \right|$$

$$\times\ \mathcal{H}^{N-1}(f_i \cap \pi_{i-1}^{-1}(U))$$

$$= \|\partial N_i\|(\pi_{i-1}^{-1}(U)),$$

from which we conclude that the second equation in (3.28) also holds for i, j finite.

Step 7: Weak convergence for normal currents.

For a discussion about weak convergence for normal currents we refer the reader to [Lan11, Sec. 5]. The main point is that on sets of normal currents for which one has uniform bounds on the masses and the masses of the boundaries (e.g. the situation in **Step 6**) the weak topology is metrizable. Concretely, consider the following set of "formal" N-forms:

(3.65) $\quad \Omega = \{\omega = f_0\, df_1 \wedge \cdots \wedge df_N$ where $f_\alpha : Z \to \mathbb{R}$ are 1-Lipschitz,

$$\mathrm{spt}\, f_\alpha \text{ is bounded and } |f_0| \leq 1\};$$

to show that $\{N_i\}_{i \in I}$ converges it suffices to show that, for each $\omega \in \Omega$, $\{N_i(\omega)\}_{i \in I}$ is a Cauchy sequence. On Ω we introduce (pseudo)distances:

$$\|\omega - \omega'\|_\Omega = \max_{\alpha=0,\cdots,N} \|f_\alpha - f'_\alpha\|_\infty$$

(3.66)

$$\|\omega - \omega'\|_{\Omega,Y} = \max_{\alpha=0,\cdots,N} \sup_{y \in Y} |(f_\alpha - f'_\alpha)(y)| \quad (Y \subset X),$$

and finally define the "support" $\mathrm{spt}\,\omega$ of ω to be $\mathrm{spt}\,f_0$.

We will use [Lan11, Thm. 5.2]: there is a $C(N)$ such that if T is an N-dimensional normal current:

(3.67) $\qquad |T(\omega) - T(\omega')| \le C(N)\|\omega - \omega'\|_{\Omega,\mathrm{spt}\,T}(\|T\|(\mathrm{spt}\,\omega) + \|\partial T\|(\mathrm{spt}\,\omega)),$

whenever $\omega, \omega' \in \Omega$. Fix $i, j \in I \cup \{\infty\}$ and $\omega \in \Omega$. We let (which is only well-defined on X_i) $\pi^*_{i,j}\omega$ denote the pull-back of ω (restricted on X_j), i.e.:

(3.68) $\qquad \pi^*_{i,j}\omega = f_0 \circ \pi_{i,j}\, df_1 \circ \pi_{i,j} \wedge \cdots \wedge df_N \circ \pi_{i,j};$

by (3.24) we have:

(3.69) $$\|\omega - \pi^*_{i,j}\omega\|_{\Omega,X_i} = O(m^{-j})$$

where the constants hidden in the $O(\cdot)$-notation are uniform in i, j. As

(3.70) $$|N_{i+1}(\omega) - N_i(\omega)| = |N_{i+1}(\omega) - N_{i+1}(\pi^*_i\omega)|,$$

combining (3.67), (3.69) with the uniform bounds on $\|N_i\|(\mathrm{spt}\,\omega)$ and $\|\partial N_i\|(\mathrm{spt}\,\omega)$ given by `Step 6`, we conclude that $\{N_i(\omega)\}_{i \in I}$ is Cauchy and thus the limit N_∞ exists.

`Step 8:` `Proof of (3.26), (3.27) and (3.28) when` $i = \infty$.

To show (3.26) we use that N_∞ is normal (i.e. the joint continuity (2.42)) to deduce:

(3.71) $$N_\infty(\pi^*_{\infty,j}\omega) = \lim_{k \to \infty} N_\infty(\pi^*_{k,j}\omega),$$

where for each k $\pi^*_{k,j}\omega$ has been extended from X_k to Z using MacShane's Lemma on each f_α; we then use the definition of the weak topology and (3.69):

$$N_\infty(\pi^*_{\infty,j}\omega) = \lim_{k \to \infty} \sup_{i \ge k} N_i(\pi^*_{k,j}\omega)$$

(3.72)
$$= \lim_{k \to \infty} \sup_{i \ge k}(N_i(\pi^*_{i,j}\omega) + O(m^{-k}))$$
$$= \lim_{k \to \infty}(N_j(\omega) + O(m^{-k}))$$
$$= N_j(\omega).$$

We now prove (3.28); fix $U \Subset X_j$ and $\varepsilon > 0$; find $\{\omega_p\}_{p=1}^{S(\varepsilon)} \subset \Omega$ such that the sets $\{\mathrm{spt}\,\omega_p\}_{p=1}^{S(\varepsilon)}$ are pairwise disjoint, $\mathrm{spt}\,\omega_p \cap X_j \subset U$ and

(3.73) $$\|N_j\|(U) \le \sum_{p=1}^{S(\varepsilon)} N_j(\omega_p) + \varepsilon.$$

As the sets $\{\mathrm{spt}\,\pi^*_{\infty,j}\omega_p \cap X_\infty\}_{p=1}^{S(\varepsilon)}$ are pairwise disjoint and:

(3.74) $$\mathrm{spt}\,\pi^*_{\infty,j}\omega_p \cap X_\infty \subset \pi^{-1}_{\infty,j}(U),$$

we have

$$
\|N_j\|(U) \leq \sum_{p=1}^{S(\varepsilon)} N_\infty(\pi_{\infty,j}^* \omega_p) + \varepsilon
$$

(3.75)

$$
\leq \|N_\infty\|(\pi_{\infty,j}^{-1}(U)) + \varepsilon,
$$

which establishes

(3.76) $$\|N_j\|(U) \leq \|N_\infty\|(\pi_{\infty,j}^{-1}(U)).$$

We now find a finite subcomplex L of $X_j^{(k)}$ (where k depends on ε) with $U \subset L$ and:

(3.77) $$\|N_j\|(L \setminus U) \leq \varepsilon.$$

For each gallery-connected component L_a of L, the argument of **Step 7** applied to the inverse system $\{\pi_{i,j}^{-1}(L_a)\}_{i \in I}$ shows that:

(3.78) $$N_i \llcorner \pi_{i,j}^{-1}(L) \to N_\infty \llcorner \pi_{\infty,j}^{-1}(L).$$

By lower semicontinuity of the mass:

$$
\|N_\infty\|(\pi_{\infty,j}^{-1}(U)) \leq \|N_\infty \llcorner \pi_{\infty,j}^{-1}(L)\|(Z)
$$

(3.79)

$$
\leq \liminf_{i \to \infty} \|N_i \llcorner \pi_{i,j}^{-1}(L)\|(Z)
$$

$$
= \|N_j\|(L)
$$

$$
\leq \|N_j\|(U) + \varepsilon.
$$

Thus (3.76), (3.79) give

(3.80) $$\|N_\infty\|(\pi_{\infty,j}^{-1}(U)) = \|N_j\|(U),$$

and a similar argument establishes:

(3.81) $$\|\partial N_\infty\|(\pi_{\infty,j}^{-1}(U)) = \|\partial N_j\|(U).$$

We now prove (3.27). Note that by (3.28), as $\mu_j \to \mu_\infty$ we have:

(3.82) $$\|N_\infty\| \leq \mu_\infty,$$

and thus [Sch14a, Thm. 1.4] gives:

(3.83) $$N_\infty = \lambda D_{\infty,1} \wedge \cdots \wedge D_{\infty,N} \, \mu_\infty;$$

concretely, λ is, up to a sign, the derivative $\frac{d\|N_\infty\|}{d\mu_\infty}$ and so:

(3.84) $$|\lambda| \leq 1.$$

Fix $i \in I$ and $\sigma \in \mathrm{Cell}_N(X_i)$; then by (3.83) we have:

(3.85) $$(N_\infty \llcorner \pi_{\infty,i}^{-1}(\sigma)) \llcorner dx_2 \wedge \cdots \wedge dx_N = \lambda \chi_{\pi_{\infty,i}^{-1}(\sigma)} D_{\infty,1} \, \mu_\infty.$$

Using (3.26) we have:

(3.86) $$\pi_{\infty,i\#}(N_\infty \llcorner \pi_{\infty,i}^{-1}(\sigma) \llcorner dx_2 \wedge \cdots \wedge dx_N) = (N_i \llcorner \sigma) \llcorner dx_2 \wedge \cdots \wedge dx_N,$$

which combined with (3.83) yields:

(3.87) $$\pi_{\infty,i\#}(\lambda \chi_{\pi_{\infty,i}^{-1}(\sigma)} D_{\infty,1} \, \mu_\infty) = \chi_\sigma D_{i,1} \, \mu_i;$$

using the disintegration theorem, (3.23) and (3.84) we conclude that $\lambda = 1 \ \mu_\infty$-a.e. on $\pi_{\infty,i}^{-1}(\sigma)$. $\qquad\square$

4. The Examples

In this Section we discuss how to use Theorem 3.20 to obtain N-dimensional normal currrents whose supports are purely 2-unrectifiable. The idea is inspired by the repeated use of branched covers, akin to what happens for Pontryagin surfaces (see [BL07]). However, these examples are topologically distinct from Pontryagin surfaces, e.g. the cohomology groups are different. Before proving the 2-unrectifiability we show that the 2-dimensional versions of these examples do not contain embedded surfaces (Thm. 4.9), in sharp contrast with the case of Carnot groups which contain Hölder surfaces.

Construction 4.1 (2-Branched covers of $[0,1]^N$)**.** Fix $1 \leq \alpha < \beta \leq N$ and let $q_{\alpha,\beta} : [0,1]^N \to [0,1]^2$ denote the projection on the (α, β)-plane. We subdivide $[0,1]^N$ into 5^N equal cells to get a cube-complex σ_{old} and we subdivide $[0,1]^2$ into 5^2 equal cells to get a square-complex Σ_{old}. Subdivide Σ_{old} into the following square-complexes which have disjoint interiors:

(1) Σ_{c} corresponding to the central square; let $\sigma_{\mathrm{c}} = q_{\alpha,\beta}^{-1}(\Sigma_{\mathrm{c}})$;
(2) Σ_{a} corresponding to the middle ring of squares; let $\sigma_{\mathrm{a}} = q_{\alpha,\beta}^{-1}(\Sigma_{\mathrm{a}})$;
(3) Σ_{o} corresponding to the outer ring of squares; let $\sigma_{\mathrm{o}} = q_{\alpha,\beta}^{-1}(\Sigma_{\mathrm{o}})$.

As $\pi_1(\sigma_{\mathrm{a}}) = \mathbb{Z}$ we let $\tilde{\sigma}_{\mathrm{a}}$ be the cube-complex which is the double cover of σ_{a}, and let σ_{new} be the cube-complex obtained by gluing $\tilde{\sigma}_{\mathrm{a}}$, σ_{c} and σ_{o} so that any two points on the boundary of $\tilde{\sigma}_{\mathrm{a}}$ which map to the same point of σ_{a} are identified. The covering map:

$$(4.2) \qquad\qquad\qquad\qquad \tilde{\sigma}_{\mathrm{a}} \to \sigma_{\mathrm{a}}$$

induces an open, cellular, surjective map:

$$(4.3) \qquad\qquad\qquad\qquad \phi_{\alpha,\beta} : \sigma_{\mathrm{new}} \to \sigma_{\mathrm{old}}$$

which restricts to an isometry on each face of σ_{new}.

Whenever $\sigma_{\mathrm{old}} \simeq [0, 5^{-i}]^N$ one can similarly apply a rescaled version of the previous construction to obtain σ_{new}.

Given a measure μ_{old} on σ_{old} which is a multiple of Lebesgue measure, one can evenly split μ_{old} across the N-cells of σ_{new} that project to the same cell of σ_{old} to obtain μ_{new} which satisfies $\phi_{\alpha,\beta\#}\mu_{\mathrm{new}} = \mu_{\mathrm{old}}$

Construction 4.4 (The examples)**.** Let X_0 be an orientable cube-complex which is the union of its N-cells which are all isometric to $[0,1]^N$. We will also assume that there is a doubling measure μ_0 on X_0 which restricts to a multiple of Lebesgue measure on each element of $\mathrm{Cell}_N(X_0)$ and such that (X_0, μ_0) is a $(1,1)$-PI space.

Let $I = \{0\} \cup \mathbb{N}$ and for $i \geq 1$ choose $1 \leq \alpha_i < \beta_i \leq N$ such that each possible pair (α, β) in the 2-ordered combinations of $\{1, \cdots, N\}$ occurss i.o.

We obtain an inverse system if we construct (X_{i+1}, μ_{i+1}) by applying the branched covering construction 4.1 to each $\sigma \in \mathrm{Cell}_N(X_i)$ using the plane (α_i, β_i).

Theorem 4.5 (The examples are AIS and WAIS)**.** *The system $\{(X_i, \mu_i)\}$ is both AIS and WAIS; moreover each weak tangent Y of the inverse limit is, up to a dilating factor in $[1/5, 1)$, the inverse limit of a system obtained using Construction 4.4.*

Proof. Step 1: Admissibility.
The only assertion that requires justification is **(IBGeom)**, as the other axioms follow from the recursive construction. Fix $x_i \in X_i$ and for $j \leq i$ let $x_j = \pi_{i,j}(x_i)$.

In passing from x_j to x_{j+1} the cardinality of the link can increase only under the following circumstance: if $x_j \in \sigma$, where $\sigma \in \mathrm{Cell}_N(X_j)$, and if

$$(4.6) \qquad\qquad q_{\alpha_j, \beta_j} : \sigma \to \mathbb{R}^2$$

denotes the projection on the (α_j, β_j)-plane, one must have

$$(4.7) \qquad\qquad q_{\alpha_j, \beta_j}(x_j) \in \partial\Sigma_{\mathrm{a}},$$

using the notation of Construction 4.1. Note that for cubes σ, σ' of different generations k, k', if $(\alpha_k, \beta_k) = (\alpha_{k'}, \beta_{k'})$ the corresponding $\partial\Sigma_{\mathrm{a}}$ and $\partial\Sigma'_{\mathrm{a}}$ are disjoint, and so if (4.7) occurrs for a given pair (α_j, β_j), it can occurr only once for that pair. In that case, the cardinality of the link can at most double; as there are at most $\binom{N}{2}$ distinct (α, β)-pairs, **(IBGeom)** follows.

`Step 2:  Weak tangents`

Assuming $(\lambda_n X_\infty, p_n, \mu_n) \to (Y_\infty, q, \nu_\infty)$, up to rescaling the metric on Y_∞ by a dilating factor in $[1/5, 1)$, we can assume that $\lambda_n = 5^{m_n}$. Choose compatible systems of basepoints $\{p_{n,k}\}$ so that

$$(4.8) \qquad\qquad \pi_{\infty,k}(p_n) = p_{n,k}.$$

A compactness argument shows that, for each $k \in I$ $(\lambda_n X_{k+m_n}, p_{n,k+m_n}, \mu_{n,k+m_n})$ subconverges to some (Y_k, q_k, ν_k), and the $\{(Y_k, \nu_k)\}_{k \in I}$ form an AIS/WAIS whose inverse limit is (Y_∞, ν_∞). $\qquad\square$

In the following it will be useful to consider the case of Construction 4.4 where X_0 is 2-dimensional, and where the branching is applied for ∞-many values of i, but not necessarily all values of i; in this case we will use $\{Y_i\}_{i \in I}$ (omitting the measures) to denote the inverse system and Y_∞ the inverse limit.

Theorem 4.9. *There is no embedding:*

$$(4.10) \qquad\qquad j : \bar{D}^2 = [0,1]^2 \to Y_\infty,$$

where $\bar{D}^2$ is the 2-dimensional closed disk.

Proof. We will argue by contradiction.

`Step 1:  `$j(\bar{D}^2)$` has nonempty interior.`

Assume that for each $k \in \mathbb{N}$ $\pi_{\infty,k}(j(\bar{D}^2))$ does not contain an 2-cell of Y_k; then $\pi_{\infty,k}(j(\bar{D}^2))$ can be retracted to the 1-skeleton of Y_k, and by (3.10) the retraction can be chosen to produce a 1-dimensional simplicial complex which is an $O(m^{-k})$-approximation of $j(\bar{D}^2)$. Letting $k \to \infty$ we conclude that $j(\bar{D}^2)$ has topological dimension 1, which contradicts that $\bar{D}^2$ has topological dimension 2.

Fix k_0 and σ such that $\sigma \in \mathrm{Cell}_2(Y_{k_0})$, $\sigma \subset \pi_{\infty,k_0}(j(\bar{D}^2))$. As in Construction 4.1 we let

$$(4.11) \qquad\qquad \tilde{\sigma} = \tilde{\sigma}_{\mathrm{c}} \cup \tilde{\sigma}_{\mathrm{a}} \cup \tilde{\sigma}_{\mathrm{o}} = \pi_{k_0}^{-1}(\sigma);$$

as $\pi_{k_0}|\tilde{\sigma}_{\mathrm{c}} \cup \tilde{\sigma}_{\mathrm{o}}$ is injective we deduce

$$(4.12) \qquad\qquad \tilde{\sigma}_{\mathrm{c}} \cup \tilde{\sigma}_{\mathrm{o}} \subset \pi_{\infty,k_0+1}(j(\bar{D}^2)).$$

Were

$$(4.13) \qquad\qquad \tilde{\sigma}_{\mathrm{a}} \setminus \pi_{\infty,k_0+1}(j(\bar{D}^2)) \neq \emptyset,$$

we could retract via a map r the set:

$$(4.14) \qquad\qquad \tilde{\sigma}_{\mathrm{a}} \cap \pi_{\infty,k_0+1}(j(\bar{D}^2))$$

to $\partial\tilde{\sigma}_{\mathrm{a}}$ while keeping $Y_{k_0+1} \setminus \tilde{\sigma}_{\mathrm{a}}$ fixed, and then $(r \circ \pi_{\infty,k_0+1})^{-1}(\tilde{\sigma}_{\mathrm{c}})$ and $(r \circ \pi_{\infty,k_0+1})^{-1}(Y_{k_0+1} \setminus \tilde{\sigma}_{\mathrm{c}})$ would disconnect $j(\bar{D}^2)$, a contradiction. Hence:

$$(4.15) \qquad\qquad \tilde{\sigma} \subset \pi_{\infty,k_0+1}(j(\bar{D}^2)).$$

We can iterate the previous argument on each cell of $\tilde{\sigma}^{(1)}$ and continue to conclude that

$$(4.16) \qquad\qquad \pi_{\infty,k_0}^{-1}(\sigma) \subset j(\bar{D}^2),$$

showing that $j(\bar{D}^2)$ has nonempty interior.

Step 2: Simple connectedness.

As any point of $\bar{D}^2$ has a simply-connected neighbourhood, it suffices to show that each non-empty open $U \subset Y_\infty$ is not simply-connected. In fact, for some k, there is a loop γ in U such that $\pi_{\infty,k} \circ \gamma$ is a generator of the fundamental group of $\partial\sigma$, where $\sigma \in \mathrm{Cell}_2(X_k)$. But then $\pi_{\infty,k+1} \circ \gamma$ is not homotopically trivial in Y_{k+1} by construction of the branched cover. $\qquad\square$

Theorem 4.17. X_∞ *is purely* 2-*unrectifiable.*

Proof. Step 1: Reduction to 2 dimensions.

We argue by contradiction assuming that there are a compact set K and a Lipschitz map f:

$$(4.18) \qquad\qquad f : K \subset \mathbb{R}^2 \to X_\infty$$

with $\mathcal{H}^2(f(K)) > 0$. Without loss of generality we can assume that $\inf I = 0$ and $X_0 = [0,1]^N$.

As k varies in I consider 1-Lipschitz functions

$$(4.19) \qquad\qquad \psi : X_k \to \mathbb{R} :$$

the collection $\{\pi_{\infty,k}^*\psi\}_{k,\psi}$ uniformly separates points in X_∞; concretely, given $x, x' \in X_\infty$ one can find $k \in I$ and a 1-Lispchitz

$$(4.20) \qquad\qquad \psi : X_k \to \mathbb{R}$$

such that:

$$(4.21) \qquad\qquad \frac{1}{2}d_{X_\infty}(x,x') \le \psi(\pi_{\infty,k}(x)) - \psi(\pi_{\infty,k}(x'));$$

in fact, we know from (3.30) that:

$$(4.22) \qquad\qquad \left|d_{X_\infty}(x,x') - (\pi_{\infty,k}^* d_{X_k})(x,x')\right| = O(m^{-k}).$$

By the Stone-Weierstrass Theorem for Lipschitz algebras [Wea99, Thm. 4.1.8] the unital algebra generated by $\{\pi_{\infty,k}^*\psi\}_{k,\psi}$ is weak* dense in $\mathrm{Lip}_{\mathrm{b}}(X_\infty)$. Moreover, each cell of X_k is isometric to a cube in $\mathbb{R}^N$ and so the differential $d\pi_{\infty,k}^*\psi$ is a linear combination of $dx_1, \cdots, dx_N$ (we abuse notation and just write x_α for $x_\alpha \circ \pi_{\infty,0}$). Thus, any derivation $D \in \mathfrak{X}(\nu)$, ν being a Radon measure on X_∞, is determined by $\{Dx_\alpha\}_{\alpha=1,\cdots,N}$.

Let ∂_1, ∂_2 be the standard basis of $\mathfrak{X}(\mathcal{L}^2 \llcorner K)$; as $\mathcal{H}^2(f(K)) > 0$ the area formula [AK00b, Thm. 8.2] implies that the derivations $f_\#\partial_1, f_\#\partial_2$ must be independent, and so there must be $\alpha < \beta$ and a compact $\tilde{K} \subset K$ with $\mathcal{L}^2(\tilde{K}) > 0$ such that

$$(4.23) \qquad\qquad \det \begin{pmatrix} \partial_1(f \circ x_\alpha) & \partial_1(f \circ x_\beta) \\ \partial_2(f \circ x_\alpha) & \partial_2(f \circ x_\beta) \end{pmatrix}$$

is uniformly bounded away from 0 on $\tilde{K}$.

For each $k \in I$ let Z_k be the cube-complex obtained by collapsing each cell of X_k in the directions $\{x_\gamma\}_{\gamma \neq \alpha, \beta}$. The $\{Z_k\}_{k \in I}$ form a 2-dimensional AIS/WAIS as the $\{Y_k\}$ discussed before Theorem 4.9. In particular, if $q_{\alpha\beta}$ denotes projection on the (α, β) plane, we have commutative diagrams:

$$
\begin{array}{ccc}
X_\infty & \xrightarrow{\;\psi_\infty\;} & Z_\infty \\
\pi_{\infty,i}\downarrow & & \downarrow \tilde{\pi}_{\infty,i} \\
X_i & \xrightarrow{\;\psi_i\;} & Z_i \\
\pi_{i,0}\downarrow & & \downarrow \tilde{\pi}_{i,0} = q_{\alpha\beta} \\
[0,1]^N & \xrightarrow{\;\psi_0\;} & [0,1]^2
\end{array}
$$

(4.24)

where the functions $\{\psi_i\}$ are 1-Lipschitz.

Step 2: Blow-up.

Consider:

$$
\begin{array}{ccc}
\tilde{K} & \xrightarrow{\;\psi_\infty \circ f\;} & Z_\infty \\
 & \searrow^{q_{\alpha\beta} \circ \psi_\infty \circ f} & \downarrow q_{\alpha\beta} \quad ; \\
 & & [0,1]^2
\end{array}
$$

(4.25)

blowing up $\psi_\infty \circ f$ at a density point of $\tilde{K}$ we obtain a diagram:

$$
\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{\;G\;} & Y_\infty \\
 & \searrow^{q_{\alpha\beta} \circ G} & \downarrow q_{\alpha\beta} \quad , \\
 & & \mathbb{R}^2
\end{array}
$$

(4.26)

where $q_{\alpha\beta} \circ G$ must be linear; by the bounds on (4.23) $q_{\alpha\beta} \circ G$ is non-singular, so taking $\bar{D}^2 \subset \mathbb{R}^2$

(4.27)
$$G : \bar{D}^2 \to Y_\infty$$

gives an embedding contradicting Theorem 4.9 (one can also argue that $q_{\alpha\beta}$ is not injective). $\qquad \square$

5. Approximation by cubical currents

In this Section we discuss how to use currents associated to cube complexes to approximate normal currents (sitting in l^∞) in the flat and weak topologies.

Definition 5.1. Let X be an N-dimensional finite cube-complex. A **cubical current** (supported on X) is a normal current of the form:

(5.2)
$$\sum_{\sigma \in \mathrm{Cell}_N(X)} \mathrm{weight}(\sigma)[\![\sigma]\!].$$

Theorem 5.3. *Let T be a compactly-supported normal current in l^∞. For each $\varepsilon > 0$ there is a compactly-supported cubical current T_c such that:*

$$(5.4) \qquad \qquad \text{Flat}\,(T - T_c) \leq \varepsilon$$

$$(5.5) \qquad \qquad \mathbf{N}\,(T_c) \leq \mathbf{N}\,(T) + \varepsilon$$

$$(5.6) \qquad \qquad \text{spt}\,T_c \subset B(\text{spt}\,T, \varepsilon).$$

In particular, T can be approximated in the weak topology by a sequence of cubical currents $\{T_n\}$ with:

$$(5.7) \qquad \qquad \lim_{n \to \infty} \mathbf{N}\,(T_n) = \mathbf{N}\,(T)$$

$$(5.8) \qquad \qquad \lim_{n \to \infty} \sup_{y \in \text{spt}\,T_n} \inf_{z \in \text{spt}\,T} \|y - z\|_\infty = 0.$$

Remark 5.9. In Theorem 5.3 one can replace l^∞ with a Banach space having the *bounded approximation property*; the only difference is the bound (5.6) which worsens to:

$$(5.10) \qquad \qquad \mathbf{N}\,(T_c) \leq C\mathbf{N}\,(T) + \varepsilon,$$

C being the norm of the projections used in the bounded approximation property.

Proof of Theorem 5.3. `Step 1:  Choice of a projection.`
    The space l^∞ has the bounded approximation property where projections can be taken to have norm 1 (see [PS12, Lem. 5.7]). Concretely, for each parameter $\varepsilon_{\text{pj}} > 0$ we can choose a norm-1 projection

$$(5.11) \qquad \qquad \pi : l^\infty \to \mathcal{S},$$

$\mathcal{S}$ begin a finite-dimensional hyperplane in l^∞, such that:

$$(5.12) \qquad \qquad \pi(\text{spt}\,T) \subset B(\text{spt}\,T, \varepsilon_{\text{pj}})$$

$$(5.13) \qquad \qquad \sup_{y \in \text{spt}\,T} \|\pi(y) - y\|_\infty \leq \varepsilon_{\text{pj}}.$$

Let

$$(5.14) \qquad \begin{aligned} H : [0,1] \times \text{spt}\,T &\to l^\infty \\ (t, x) &\mapsto (1 - t)x + t\pi(x); \end{aligned}$$

by the *Homotopy formula* (see [Fed69, 4.1.9, 4.1.10]: the extension to normal metric currents is straightforward)

$$(5.15) \qquad \partial H_\#([0,1] \times T) = H(1, \cdot)_\# T - H(0, \cdot)_\# T - H_\#([0,1] \times \partial T),$$

where if

$$(5.16) \qquad T = \sum_\alpha D_{\alpha_1} \wedge \cdots \wedge D_{\alpha_N} \,\|T\|,$$

we let:

$$(5.17) \qquad [0,1] \times T = \sum_\alpha \partial_t \wedge D_{\alpha_1} \wedge \cdots \wedge D_{\alpha_N} \,\mathcal{L}^1 \,\llcorner\, [0,1] \times \|T\|.$$

We let $T_{\text{pj}} = H(1, \cdot)_\# T$, and note that by (5.13) and (5.15) we have:

$$(5.18) \qquad \text{Flat}\,(T_{\text{pj}} - T) \leq \varepsilon_{\text{pj}} \mathbf{N}\,(T)\,;$$

as $H(1, \cdot) = \pi$ we conclude that:

$$(5.19) \qquad\qquad \mathbf{N}(T_{\mathrm{pj}}) \leq \mathbf{N}(T)$$

$$(5.20) \qquad\qquad \mathrm{spt}\, T_{\mathrm{pj}} \subset B(\mathrm{spt}\, T, \varepsilon_{\mathrm{pj}}) \cap \mathcal{S}.$$

Step 2: Smoothing T_{pj}.

T_{pj} can be regarded as a normal current on the finite-dimensional vector space $\mathcal{S}$; by a standard smoothing argument ([Fed69, 4.1.18]) for each $\varepsilon_{\mathrm{sm}} > 0$ we can find T_{sm} with support in $\mathcal{S}$ such that:

$$(5.21) \qquad\qquad \mathrm{Flat}\,(T_{\mathrm{sm}} - T_{\mathrm{pj}}) \leq \varepsilon_{\mathrm{sm}}$$

$$(5.22) \qquad\qquad \mathrm{spt}\, T_{\mathrm{sm}} \subset B(T_{\mathrm{pj}}, \varepsilon_{\mathrm{sm}})$$

$$(5.23) \qquad\qquad \mathbf{N}(T_{\mathrm{sm}}) \leq \mathbf{N}(T_{\mathrm{pj}})$$

$$(5.24) \qquad\qquad \|T_{\mathrm{sm}}\| \ll \mathcal{L}^{\dim \mathcal{S}}$$

$$(5.25) \qquad\qquad \|\partial T_{\mathrm{sm}}\| \ll \mathcal{L}^{\dim \mathcal{S}},$$

where we use $\mathcal{L}^{\dim \mathcal{S}}$ to denote Lebesgue measure on $\mathcal{S}$.

Let $\mathbb{Z}^{\dim \mathcal{S}}$ denote the integer lattice in $\mathcal{S}$; because of equations (5.24), (5.25), which express absolute continuity of $\|T_{\mathrm{sm}}\|$ and $\|\partial T_{\mathrm{sm}}\|$ wrt. the Lebesgue measure, we can approximate these currents in the *flat norm* by cubical currents: i.e. for each $\varepsilon_{\mathrm{cb}} > 0$ we can find cubical currents T_{cb}, $T_{\partial \mathrm{cb}}$ and $n_0 \in \mathbb{N}$ with $2^{-n_0} \leq \varepsilon_{\mathrm{cb}}$ and where:

> **(Sm1):** The supports of T_{cb} and $T_{\partial \mathrm{cb}}$ belong, respectively, to the N-skeleton and the $(N-1)$-skeleton of the cubulation of $\mathcal{S}$ associated to $2^{-n_0}\mathbb{Z}^{\dim \mathcal{S}}$;
> **(Sm2):** T_{cb} is N-dimensional and $T_{\partial \mathrm{cb}}$ is $(N-1)$-dimensional and one has:

$$(5.26) \qquad\qquad \mathbf{M}(T_{\mathrm{cb}}) \leq \mathbf{M}(T_{\mathrm{sm}})$$

$$(5.27) \qquad\qquad \mathbf{M}(T_{\partial \mathrm{cb}}) \leq \mathbf{M}(\partial T_{\mathrm{sm}})$$

$$(5.28) \qquad \mathrm{Flat}\,(T_{\mathrm{cb}} - T_{\mathrm{sm}}) + \mathrm{Flat}\,(T_{\partial \mathrm{cb}} - \partial T_{\mathrm{sm}}) \leq \varepsilon_{\mathrm{cb}}$$

$$(5.29) \qquad\qquad \mathrm{spt}\, T_{\mathrm{cb}} \cup \mathrm{spt}\, T_{\partial \mathrm{cb}} \subset B(\mathrm{spt}\, T_{\mathrm{sm}}, \varepsilon_{\mathrm{cb}}).$$

Step 3: Application of the Deformation Theorem in $\mathcal{S}$.

The argument is then completed applying the Federer-Fleming Deformation Theorem along the lines of [Fed69, 4.2.23, 4.2.24]. Concretely, (5.28) gives:

$$(5.30) \qquad\qquad \mathrm{Flat}\,(\partial T_{\mathrm{cb}} - T_{\partial \mathrm{cb}}) \leq 2\varepsilon_{\mathrm{cb}},$$

where the flat distance can be actually computed in the metric space $B(\mathrm{spt}\, T_{\mathrm{sm}}, 3\varepsilon_{\mathrm{cb}}) \cap \mathcal{S}$, and thus there is a normal current T_{fl} supported in $\mathcal{S}$ with:

$$(5.31) \qquad \mathbf{M}(\partial T_{\mathrm{cb}} - T_{\partial \mathrm{cb}} - \partial T_{\mathrm{fl}}) + \mathbf{M}(T_{\mathrm{fl}}) \leq 3\varepsilon_{\mathrm{cb}}$$

$$(5.32) \qquad\qquad \mathrm{spt}\, T_{\mathrm{fl}} \subset B(\mathrm{spt}\, T_{\mathrm{sm}}, 3\varepsilon_{\mathrm{cb}}).$$

For any $\varepsilon_{\mathrm{df},1} > 0$, applying the Deformation Theorem we can find $n_1 \in \mathbb{N}$, an $(N-1)$-cubical current $T_{\mathrm{df},1}$ and an N-normal current $S_{\mathrm{df},1}$ such that:

> **(DF1):** The support of $T_{\mathrm{df},1}$ belongs to the $(N-1)$-skeleton of the cubulation of $\mathcal{S}$ with vertices on $2^{-n_1}\mathbb{Z}^{\dim \mathcal{S}}$, where $2^{-n_1} \leq \varepsilon_{\mathrm{df},1}$;

(DF2): Letting $C_{\mathbb{S}}$ denote a constant depending only on $\mathbb{S}$, one has:

$$(5.33) \qquad \partial T_{\mathrm{cb}} - T_{\partial\mathrm{cb}} - \partial T_{\mathrm{fl}} = T_{\mathrm{df},1} + \partial S_{\mathrm{df},1}$$

$$(5.34) \qquad \mathbf{M}\left(S_{\mathrm{df},1}\right) \leq C_{\mathbb{S}}\varepsilon_{\mathrm{df},1}\mathbf{M}\left(\partial T_{\mathrm{cb}} - T_{\partial\mathrm{cb}} - \partial T_{\mathrm{fl}}\right)$$

$$(5.35) \qquad \mathbf{M}\left(T_{\mathrm{df},1}\right) \leq C_{\mathbb{S}}\left[\mathbf{M}\left(\partial T_{\mathrm{cb}} - T_{\partial\mathrm{cb}} - \partial T_{\mathrm{fl}}\right) + \varepsilon_{\mathrm{df},1}\mathbf{M}\left(\partial T_{\partial\mathrm{cb}}\right)\right]$$

$$(5.36) \qquad \operatorname{spt} T_{\mathrm{df},1} \cup \operatorname{spt} S_{\mathrm{df},1} \subset B(\operatorname{spt} T_{\mathrm{sm}}, 3\varepsilon_{\mathrm{cb}} + 3\varepsilon_{\mathrm{df},1}).$$

Note that as $C_{\mathbb{S}}$ depends only on $\mathbb{S}$, the rhs. of (5.34) can be made arbitrarly small by choosing $\varepsilon_{\mathrm{df},1}$ sufficiently small. Note also that in applying the Deformation Theorem we used that the boundary of $T_{\partial\mathrm{cb}}$ is cubical in order to claim that $T_{\mathrm{df},1}$ is cubical.

As by (5.33) the boundary of $S_{\mathrm{df},1} + T_{\mathrm{fl}}$ is cubical, we can reiterate and for $\varepsilon_{\mathrm{df},2}$ we can find $n_2 \in \mathbb{N}$, an N-cubical current $T_{\mathrm{df},2}$ and an $(N+1)$-normal current $S_{\mathrm{df},2}$ such that:

(DF3): The support of $T_{\mathrm{df},2}$ belongs to the N-skeleton of the cubulation of $\mathbb{S}$ with vertices on $2^{-n_2}\mathbb{Z}^{\dim \mathbb{S}}$, where $2^{-n_2} \leq \varepsilon_{\mathrm{df},2}$;

(DF4): One has:

$$(5.37) \qquad S_{\mathrm{df},1} + T_{\mathrm{fl}} = \partial S_{\mathrm{df},2} + T_{\mathrm{df},2}$$

$$(5.38) \qquad \mathbf{M}\left(T_{\mathrm{df},2}\right) \leq C_{\mathbb{S}}\left[\mathbf{M}\left(S_{\mathrm{df},1} + T_{\mathrm{fl}}\right) + \varepsilon_{\mathrm{df},2}\mathbf{M}\left(\partial S_{\mathrm{df},1} + \partial T_{\mathrm{fl}}\right)\right]$$

$$(5.39) \qquad \mathbf{M}\left(S_{\mathrm{df},2}\right) \leq C_{\mathbb{S}}\varepsilon_{\mathrm{df},2}\mathbf{M}\left(S_{\mathrm{df},1} + T_{\mathrm{fl}}\right)$$

$$(5.40) \qquad \operatorname{spt} T_{\mathrm{df},2} \cup \operatorname{spt} S_{\mathrm{df},2} \subset B(\operatorname{spt} T_{\mathrm{sm}}, 4\varepsilon_{\mathrm{cb}} + 4\varepsilon_{\mathrm{df},1} + 2\varepsilon_{\mathrm{df},2}).$$

The desired current T_{c} is then obtained by letting

$$(5.41) \qquad T_{\mathrm{c}} = T_{\mathrm{cb}} - T_{\mathrm{df},2}$$

and by choosing the ε-parameters sufficiently small. $\qquad\square$

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